ACAP
Alabama Comprehensive Assessment Program
SUMMATIVE

Grade 3
Mathematics

Alabama Educator Instructional Supports

Alabama Course of Study Standards

DATA RECOGNITION DRC CORPORATION

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Table of Contents

Introduction

The *Alabama Instructional Supports: Mathematics* is a companion to the 2019 *Alabama Course of Study: Mathematics* for Grades K–12. Instructional supports are foundational tools that educators may use to help students become independent learners as they build toward mastery of the *Alabama Course of Study* content standards. **Instructional supports are designed to help educators engage their students in exploring, explaining, and expanding their understanding of the content standards.**

The content standards contained within the course of study may be accessed on the Alabama State Department of Education (ALSDE) website: https://www.alabamaachieves.org/. When examining these instructional supports, educators are reminded that content standards indicate minimum content—what all students should know and be able to do by the end of each grade level or course. Local school systems may have additional instructional or achievement expectations and may provide instructional guidelines that address content sequence, review, and remediation.

The instructional supports are organized by standard. Each standard's instructional support includes a statement of the content standard, guiding questions with connections to mathematical practices, key academic terms, and additional resources.

Content Standards

The content standards are the statements from the 2019 *Alabama Course of Study: Mathematics* that define what all students should know and be able to do at the conclusion of a given grade level or course. Content standards contain minimum required content and complete the phrase "Students will ______."

Guiding Questions with Connections to Mathematical Practices

Guiding questions are designed to create a framework for the given standards and to engage students in exploring, explaining, and expanding their understanding of the content standards provided in the 2019 *Alabama Course of Study: Mathematics*. Therefore, each guiding question is written to help educators convey important concepts within the standard. By utilizing guiding questions, educators are engaging students in investigating, analyzing, and demonstrating knowledge of the underlying concepts reflected in the standard. An emphasis is placed on the integration of the eight Student for Mathematical Practices.

The Student Mathematical Practices describe the varieties of expertise that mathematics educators at all levels should seek to develop in their students. They are based on the National Council of Teachers of Mathematics process standards and the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up: Helping Children Learn Mathematics*.

The Student Mathematical Practices are the same for all grade levels and are listed below.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Each guiding question includes a representative set of sample activities and examples that can be used in the classroom. The set of activities and examples is not intended to include all the activities and examples that would be relevant to the standard.

Key Academic Terms

These academic terms are derived from the standards and are to be incorporated into instruction by the educator and used by the students.

Additional Resources

Additional resources are included that are aligned to the standard and may provide additional instructional support to help students build toward mastery of the designated standard. Please note that while every effort has been made to ensure all hyperlinks are working at the time of publication, web-based resources are impermanent and may be deleted, moved, or archived by the information owners at any time and without notice. Registration is not required to access the materials aligned to the specified standard. Some resources offer access to additional materials by asking educators to complete a registration. While the resources are publicly available, some websites may be blocked due to Internet restrictions put in place by a facility. Each facility's technology coordinator can assist educators in accessing any blocked content. Sites that use Adobe Flash may be difficult to access after December 31, 2020, unless users download additional programs that allow them to open SWF files outside their browsers.

Printing This Document

It is possible to use this entire document without printing it. However, if you would like to print this document, you do not have to print every page. First, identify the page ranges of the standards or domains that you would like to print. Then, in the print pop-up command screen, indicate which pages you would like to print.

1

Operations and Algebraic Thinking
Represent and solve problems involving multiplication and division.
1. Illustrate the product of two whole numbers as equal groups by identifying the number of groups and the number in each group and represent as a written expression.

Guiding Questions with Connections to Mathematical Practices:

How can multiplication problems be interpreted?

M.P.7. Look for and make use of structure. Demonstrate that a multiplication problem can be interpreted as x groups of y objects; skip-counting by x a total of y times or skip-counting by y a total of x times. For example, 4×6 can be interpreted as 4 groups of 6 blocks or as 6 groups of 4 blocks. Additionally, multiplication can be interpreted through repeated addition; 4×6 can be interpreted as $6 + 6 + 6 + 6$.

- Ask students to explain how to use skip-counting to interpret a given multiplication problem.
  - 8×7

    count by 8 a total of 7 times (8, 16, 24, 32, 40, 48, 56)

    OR

    count by 7 a total of 8 times (7, 14, 21, 28, 35, 42, 49, 56)
  - 9×3

    count by 9 a total of 3 times (9, 18, 27)

    OR

    count by 3 a total of 9 times (3, 6, 9, 12, 15, 18, 21, 24, 27)

- Ask students to provide an addition problem that could be used to interpret a given multiplication problem.
    - 2×9

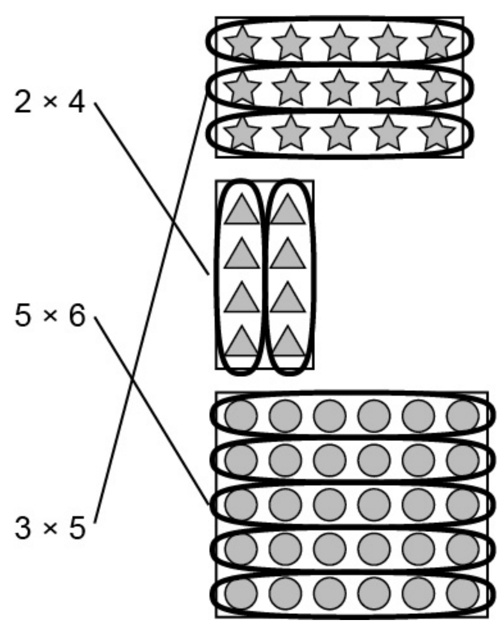
      9 + 9 or 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2
    - 7×5

      5 + 5 + 5 + 5 + 5 + 5 + 5 or 7 + 7 + 7 + 7 + 7

How can a multiplication problem be represented visually?

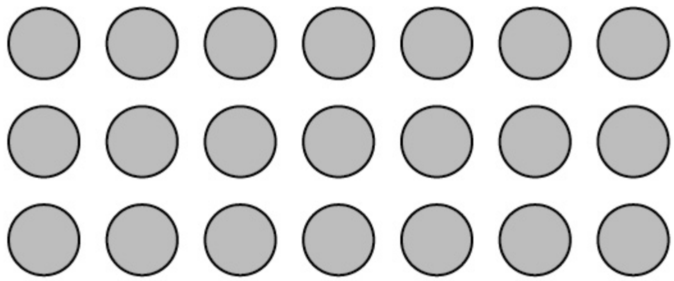
M.P.1. Make sense of problems and persevere in solving them. Create a variety of visual models to represent a multiplication problem and solve by counting. For example, the problem "Amy has 4 bags with 6 marbles in each bag. How many marbles does Amy have altogether?" can be modeled and solved by creating an array of dots in 4 rows of 6 and skip-counting by 6 to represent the 4 rows, resulting in 24 marbles. Additionally, the problem could also be represented by drawing 4 rectangles with 6 dots in each rectangle, still resulting in 24 marbles.

- Ask students to match the multiplication problem to the diagram that shows one way to interpret the problem.



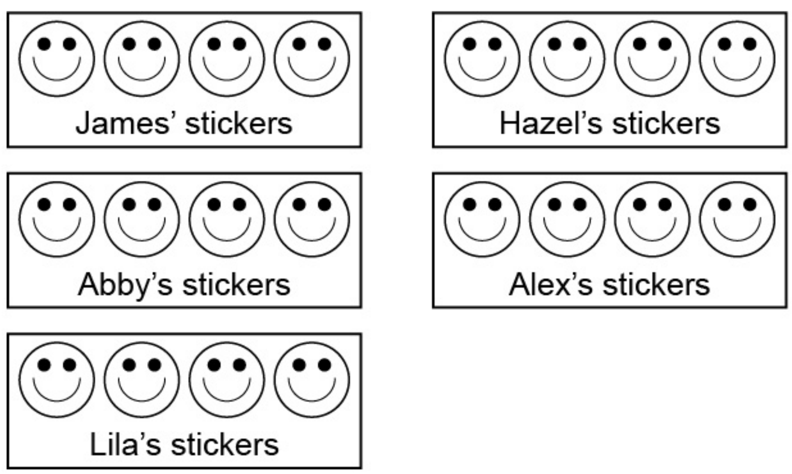
- Ask students to create a visual model that could be used to represent a given multiplication problem. Then, ask students to explain how to use the model and skip-counting to solve the multiplication problem.

3×7



I count by 7, exactly 3 times, which equals 21.

- Ask students to create a visual model that could be used to represent a scenario of x groups of y objects (or vice-versa). Then, ask students to explain how to use skip-counting to solve the multiplication problem. For example, “Elena gives 4 stickers to each of her 5 friends. How many stickers does Elena give to her friends altogether?”



I count by 4, exactly 5 times, which equals 20 stickers.

Key Academic Terms:

product, groups of equal size, row, column, skip-count, visual models

Additional Resources:

- Book: Davenport, L. R., Henry, C.S., Clements, D.H., & Sarama, J. (2019). *No more math fact frenzy*. Portsmouth, NH: Heinemann.
- Lesson: Cookie dough
- Article: Operations & algebraic thinking
- Book: Neuschwander, C. (1998). *Amanda Bean's amazing dream*. New York, NY: Scholastic Press. Activity
- Book: Pinczes, E. J. (1993). *One hundred hungry ants*. Boston, MA: Houghton Mifflin Harcourt Books for Young Readers. Activity

2

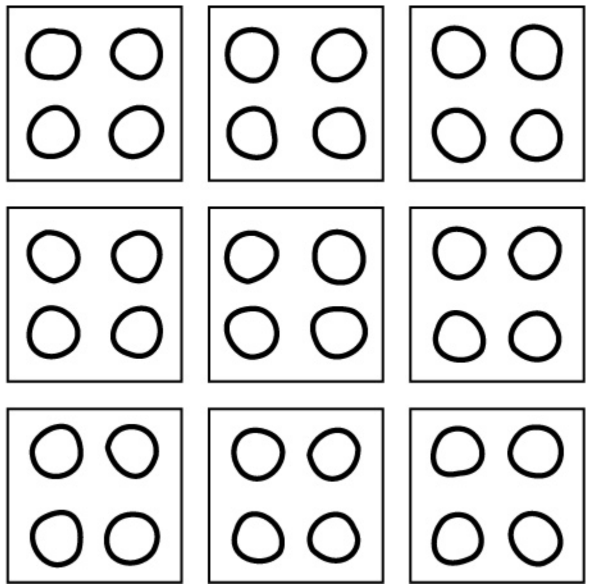
Operations and Algebraic Thinking
Represent and solve problems involving multiplication and division.
2. Illustrate and interpret the quotient of two whole numbers as the number of objects in each group or the number of groups when the whole is partitioned into equal shares.

Guiding Questions with Connections to Mathematical Practices:

How can real-world situations be modeled by division problems?

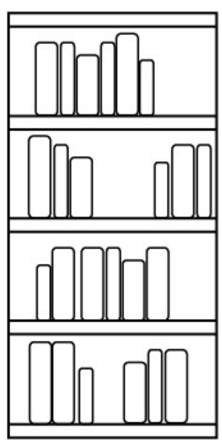
M.P.4. Model with mathematics. Demonstrate that a division expression represents either the number of objects in each group when the total number is partitioned evenly into a given number of groups or the number of groups when the total number is partitioned into groups that each contain a given number. For example, $42 \div 7$ models the number of berries in each bowl when 42 berries are divided into 7 bowls or the number of bowls when 42 berries are divided by placing 7 berries in each bowl. Additionally, real-world division problems often use the words: each, equal, evenly, or divided into.

- Ask students to illustrate, write, and then solve a division problem that could be used to represent a given real-world situation.
  - Jonas has 36 toy cars to put into garages. Each garage holds 4 cars. How many garages does he need so that each car is in a garage?



36 ÷ 4 = 9, so Jonas needs 9 garages.

  - Kayla has 24 books at her house. Her dad builds her a bookshelf with 4 shelves. If Kayla wants to divide her books evenly, how many books does she need to place on each shelf?



24 ÷ 4 = 6, so Kayla places 6 books on each shelf.

- Ask students to write two real-world situations that could be modeled by a given division problem. Ask them to write the first situation so that the quotient tells the number of objects in each group and then ask them to write the second situation so that the quotient tells the number of groups.

$$48 \div 6 = 8$$

Situation 1—Mrs. Jones has 48 markers for her students to use for an art project. She has 6 table groups in her class and wants each table group to have the same number of markers. How many markers can she give to each table group?

Situation 2—48 students in second grade are going on a field trip. The students will ride in cars that each seat 6 students. How many cars are needed to take all the students on the field trip?

Key Academic Terms:

quotient, partition, division, expression, equal share

Additional Resources:

- Book: Davenport, L. R., Henry, C.S., Clements, D.H., & Sarama, J. (2019). *No more math fact frenzy*. Portsmouth, NH: Heinemann.
- Article: Operations & algebraic thinking
- Activity: Identify the unknown
- Book: Hutchins, P. (1989). *The doorbell rang*. New York, NY: Greenwillow Books. Activity
- Lesson: Building a division model
- Lesson: Fish tanks

3

Operations and Algebraic Thinking
Represent and solve problems involving multiplication and division.
3. Solve word situations using multiplication and division within 100 involving equal groups, arrays, and measurement quantities; represent the situation using models, drawings, and equations with a symbol for the unknown number.

Guiding Questions with Connections to Mathematical Practices:

How can a word problem be interpreted to determine the unknown in a multiplication or division problem?

M.P.7. Look for and make use of structure. Understand that a word problem with an unknown product is a multiplication problem, and a word problem with an unknown number of groups or an unknown group size can be thought of as both a multiplication problem with unknown factors and as a division problem. For example, the problem "There are 15 candies to be placed in jars, with 5 in each jar. How many jars are needed?" can be represented as $15 \div 5 = \square$, $5 \times \square = 15$, or $\square \times 5 = 15$. Additionally, it is sometimes easier to solve an equation with an unknown number by rewriting it using the opposite operation; for example, $15 \div 5 = \square$ might be easier to solve than $\square \times 5 = 15$.

- Ask students to study a series of pairs of related multiplication and division equations with the same number unknown for each equation in the pair. Have the students identify which equation was easier to solve in each pair and explain why.
    - $2 \times \square = 18$ and $18 \div 2 = \square$
    - $5 \times 4 = \square$ and $\square \div 4 = 5$
    - $\square \times 9 = 54$ and $54 \div \square = 9$

  Guide students to make any generalizations about themselves as math learners. For example, some students may always think multiplication is easier to solve than division. Others may find that when the unknown is the product or quotient, that equation is the easier equation to solve. Still others may find that difficulty in solving equations depends on the actual numbers being used in each individual problem, regardless of the operation used.

- Ask students to write and solve both a multiplication equation and a division equation that could be used to represent a given word problem. For example, “In the school cafeteria, there are 3 tables for the students in Mr. Miller’s class to sit. The class has 21 students and each table must have the same number of students sitting at it. How many students can sit at each table?”

  $21 \div 3 = \square$ and $3 \times \square = 21$

  $\square = 7$ students

- Ask students to use a multiplication equation and a related division equation to write a word problem that could be represented by both given equations.

  $\square \times 8 = 48$ and $48 \div \square = 8$

  Calvin rode his bike for a total of 48 miles over 8 days. He rode the same number of miles each day. How many miles did he ride his bike each day?

How can a real-world multiplication or division problem be represented in a variety of ways?

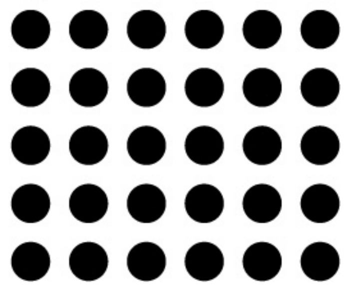
M.P.5. Use appropriate tools strategically. Represent real-world multiplication and division problems in a variety of ways. For example, to model $4 \times 8 = \square$, make 4 groups containing 8 objects, or make an array by drawing 4 rows of dots with 8 dots in each row. Additionally, when using a model to represent a multiplication or division problem, either the total, the number of groups, or the number of objects in each group serves as the unknown that the model helps to find.

- Ask students to model real-world multiplication problems using an equation, an array, and a set of objects. For example, the problem "Max reads for 5 minutes every night for 6 nights. How many total minutes does he read?" Some possible student responses are shown.

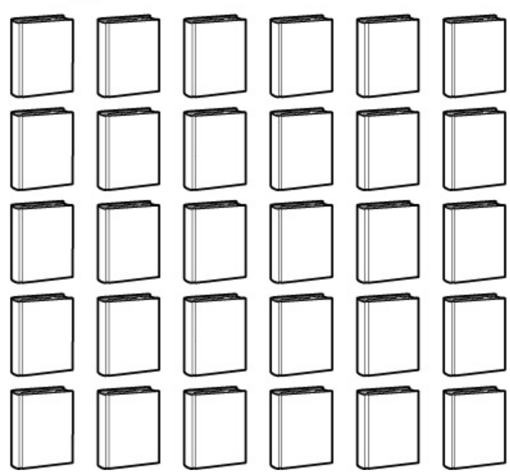
Equation:

$5 \times 6 = \square$

Array:



Drawing:

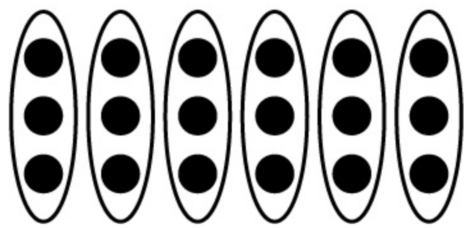


- Ask students to model real-world division problems using an equation, a set of equal groups, and a set of objects. For example, give students the problem "Charlotte scoops 18 scoops of ice cream onto 6 different ice cream cones. Each cone has the same number of scoops. How many scoops does each cone have?" Some possible student responses are shown.

Equation:

$18 \div 6 = \square$

Set of Equal Groups:



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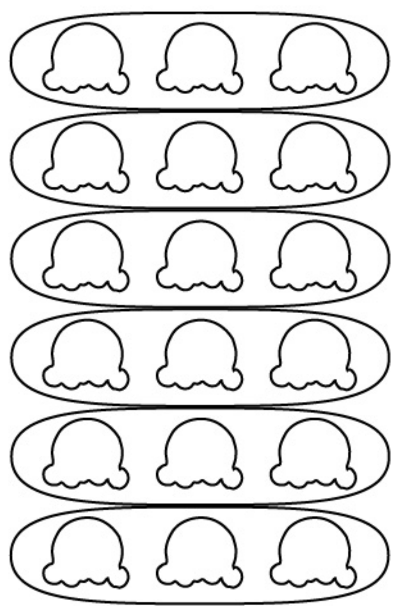


Key Academic Terms:

equal groups, array, factor, product, multiplication, division, unknown, equation, represent, measurement quantities, row, column

Additional Resources:

- Book: Davenport, L. R., Henry, C.S., Clements, D.H., & Sarama, J. (2019). *No more math fact frenzy*. Portsmouth, NH: Heinemann.
- Book: Pinczes, E. J. (1993). *One hundred hungry ants*. Boston, MA: Houghton Mifflin Harcourt Books for Young Readers. Activity
- Activity: Array picture cards
- Activity: Word problems: arrays
- Activity: Missing numbers: division
- Lesson: Cookie dough
- Lesson: Navigating road blocks in problem-solving
- Book: Giganti Jr., P. (1999). *Each orange had 8 slices*. New York, NY: Greenwillow Books. Activity

4

Operations and Algebraic Thinking
Represent and solve problems involving multiplication and division.
4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers.

Guiding Questions with Connections to Mathematical Practices:

How can multiplication facts be used to solve division problems?

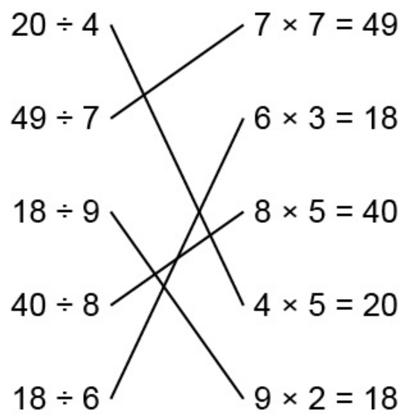
M.P.7. Look for and make use of structure. Use the relationships between multiplication and division to solve problems. For example, the result of the division expression $42 \div 7$ can be found by relating it to the multiplication equation $7 \times 6 = 42$. Additionally, another way to use multiplication to help solve the division problem $42 \div 7$ would be to answer the question: "7 times what number equals 42?"

- Ask students to name a multiplication equation that could be used to help solve a given division problem.
  - $64 \div 8$

    $8 \times 8 = 64$
  - $21 \div 3$

    $3 \times 7 = 21$

- Ask students to match the division problem to the related multiplication problem that could be used to help solve it.



- Ask students to write a number in the box in order to correctly complete a multiplication statement that would be helpful to solve a given division problem.

    - 27 ÷ 9

      "9 times ☐ is 27"

      3

    - 15 ÷ 3

      "3 times ☐ is 15"

      5

How can an unknown number in a multiplication or division problem be found?

M.P.7. Look for and make use of structure. Demonstrate that the unknown number in a multiplication or division equation is the number that makes the equation true. Use the meanings of multiplication and division and the relationship between multiplication and division to determine the unknown number. For example, $4 = 12 \div \square$ is the same as $4 \times \square = 12$. Both equations model the question "Given 4 groups, how many are in each group if the total is 12?" and can be used to determine that 3 groups of 4 are equal to 12. Additionally, once the unknown value for either a multiplication or a division equation is found, 3 other related equations can be made using the same 3 numbers for a total of 2 multiplication and 2 division equations.

- Ask students to use the meaning of multiplication or division to solve a problem. For example, when given 6×3, ask students to draw 3 sets of 6 dots and to count the total number of dots. As a further example, when given $20 \div 5$, give students 20 cards to sort into 5 stacks and ask students to count the number of cards in each stack.
- Ask students to use a related fact of the opposite operation to help identify the unknown number that makes a multiplication or a division equation true.
  - $7 \times \square = 35$

    5 because 35 ÷ 5 = 7
  - $72 \div \square = 9$

    8 because 9 × 8 = 72
  - $\square \times 2 = 12$

    6 because 12 ÷ 2 = 6
  - $\square \div 4 = 7$

    28 because 7 × 4 = 28

- Ask students to identify the unknown number that makes a multiplication or a division equation true and then use that to write the 3 other equations that are related to that given multiplication or division equation.
    - $9 \times 6 = \square$

      $\square = 54$

      $6 \times 9 = 54$

      $54 \div 6 = 9$

      $54 \div 9 = 6$
    - $21 \div \square = 3$

      $\square = 7$

      $21 \div 3 = 7$

      $3 \times 7 = 21$

      $7 \times 3 = 21$

Key Academic Terms:

unknown, multiplication, division, related equation, product

Additional Resources:

- Activity: Identify the unknown
- Activity: I Spy division
- Video: Finding unknown factors in multiplication equations

5

Operations and Algebraic Thinking
Understand properties of multiplication and the relationship between multiplication and division. *Note: Students need not use formal terms for these properties.*
5. Develop and apply properties of operations as strategies to multiply and divide.

Guiding Questions with Connections to Mathematical Practices:

How does grouping numbers to find a known fact help solve multiplication and division problems?

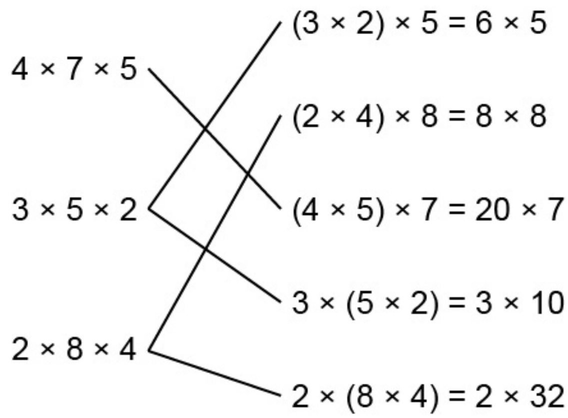
M.P.1. Make sense of problems and persevere in solving them. Group known facts in an expression. For example, $6 \times 2 \times 5$ can be reorganized using properties of operations as $6 \times (2 \times 5)$, or 6×10. Also, knowing that $9 \times 4 = 36$ means knowing that $36 \div 4 = 9$ and $36 \div 9 = 4$. Additionally, there is often more than one way in which a problem can be reorganized in order to make it easier to solve.

- Ask students to use grouping of known facts to solve multiplication problems.
  - $7 \times 4 \times 2$

    $7 \times (4 \times 2) = 7 \times 8 = 56$
  - $3 \times 6 \times 2$

    $(3 \times 2) \times 6 = 6 \times 6 = 36$

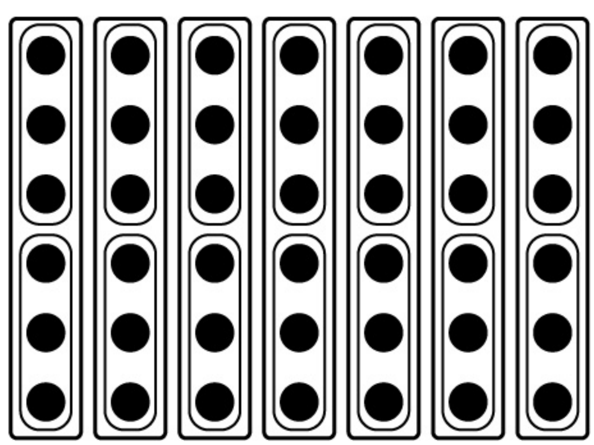
- Ask students to match an expression with equivalent forms of that expression. The equivalent forms are ones where the numbers have been reorganized or grouped in such a way that they are easier to solve.



How can numbers be decomposed to solve multiplication and division problems?

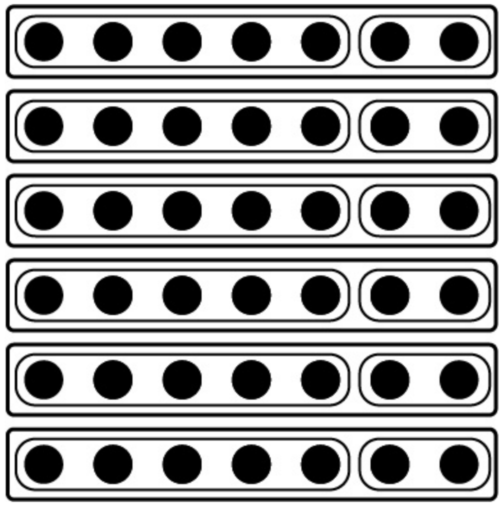
M.P.1. Make sense of problems and persevere in solving them. Decompose a multiplication expression into the sum of two multiplication expressions. For example, 9×6 can be decomposed to $9 \times (5 + 1)$, then rewritten as $(9 \times 5) + (9 \times 1)$, which is $45 + 9$. Also, $32 \div 4$ can be decomposed into 32 divided by 2 and then divided by 2 again, or $32 \div 2 = 16$ and then $16 \div 2 = 8$. Additionally, verify that there is often more than one way in which the numbers can be decomposed.

- Ask students to solve multiplication problems by decomposing the problems into the sum of two multiplication expressions.
    - 6×7



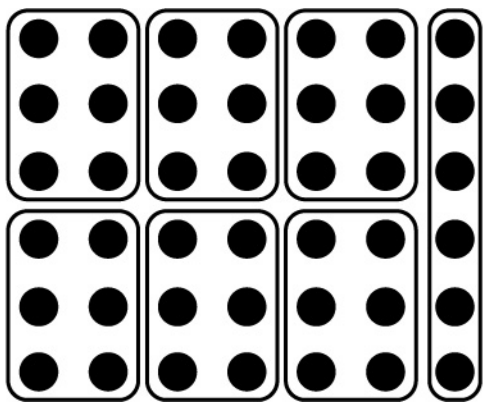
$(3 + 3) \times 7 = (3 \times 7) + (3 \times 7) = 21 + 21 = 42$

OR



$6 \times (5 + 2) = (6 \times 5) + (6 \times 2) = 30 + 12 = 42$

OR



$6 \times (6 + 1) = (6 \times 6) + (6 \times 1) = 36 + 6 = 42$

- 8×8

  $8 \times (4 \times 2) = (8 \times 4) \times 2 = 32 \times 2 = 64$

  OR

  $8 \times (4 + 4) = (8 \times 4) + (8 \times 4) = 32 + 32 = 64$

  OR

  $8 \times (5 + 3) = (8 \times 5) + (8 \times 3) = 40 + 24 = 64$

- Ask students to solve division problems using decomposition.
    - $48 \div 8$

      $48 \div (4 \times 2) = (48 \div 4) \div 2 = 12 \div 2 = 6$

      OR

      $48 \div (2 \times 2 \times 2) = (48 \div 2) \div 2 \div 2 = 24 \div 2 \div 2 = 12 \div 2 = 6$

      OR

      $48 \div 8 = (40 + 8) \div 8 = (40 \div 8) + (8 \div 8) = 5 + 1 = 6$
    - $42 \div 6$

      $42 \div (2 \times 3) = (42 \div 2) \div 3 = 21 \div 3 = 7$

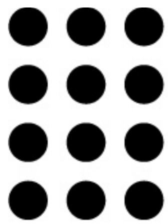
      OR

      $42 \div 6 = (30 + 12) \div 6 = (30 \div 6) + (12 \div 6) = 5 + 2 = 7$

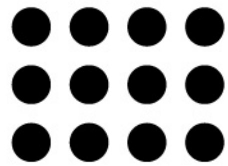
How can the properties of operations be shown using visual representations?

M.P.4. Model with mathematics. Represent how to solve multiplication and division problems using the properties of operations with models. For example, use an array model or area model to show that 7×4 is the same as $(7 \times 2) + (7 \times 2)$. Additionally, an array model can be used to show that 8×6 is the same as 6×8.

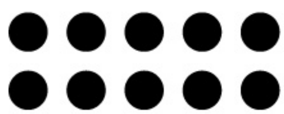
- Ask students to use an array model to illustrate the commutative property of multiplication. Some possible student responses are shown.



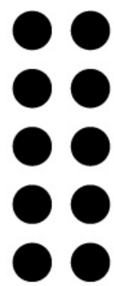
4 × 3 = 12



3 × 4 = 12



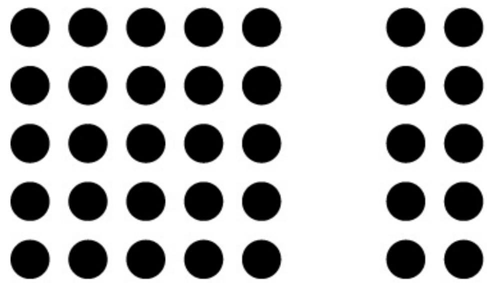
2 × 5 = 10



5 × 2 = 10

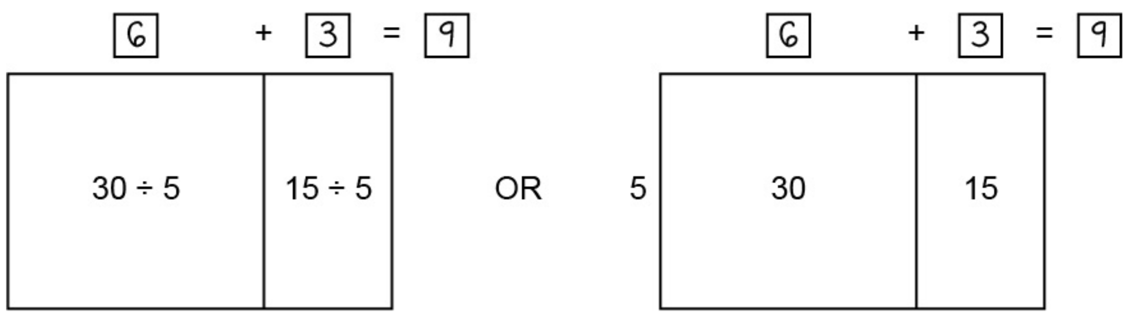
- Ask students to use an array model or an area model to illustrate solving a multiplication or division problem using the distributive property.

5×7

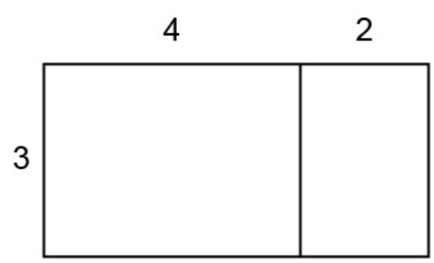


(5 × 5) + (5 × 2) = 35

$45 \div 5$



- Ask students to write and solve the equation that is being represented by the area model.



$(3 \times 4) + (3 \times 2) = 12 + 6 = 18$

OR

$3 \times (4 + 2) = 3 \times 6 = 18$

Why does reordering numbers to find a known fact help solve multiplication problems but not division problems?

M.P.3. Construct viable arguments and critique the reasoning of others. Understand that knowing the result of a multiplication expression $a \times b$ also means knowing the result of $b \times a$. Confirm that this strategy works for multiplication, but not division. For example, knowing $7 \times 3 = 21$ means also knowing $3 \times 7 = 21$. However, $21 \div 3$ is not the same as $3 \div 21$. Additionally, a visual model can help to demonstrate why this strategy works for multiplication, but not division.

- Ask students to state whether statements are true or false from a series of statements involving the commutative property.

Since $2 \times 6 = 12$ then $6 \times 2 = 12$.	*true*
Since $81 \div 9 = 9$ then $9 \div 81 = 9$.	*false*
Since $3 + 8 = 11$ then $8 + 3 = 11$.	*true*
Since $6 - 2 = 4$ then $2 - 6 = 4$.	*false*
$24 \div 4 \div 2 = 4 \div 2 \div 24$	*false*
$5 \times 9 \times 2 = 5 \times 2 \times 9$	*true*

- Ask students to study a series of equations in the forms $a \times b = c$ and $a \div b = c$ and ask them to reorder the numbers in each equation to see if the results are the same. Once students have completed that task, discuss what they observed about reordering with multiplication versus reordering with division. Guide students to see that the reordering strategy gives the same result for multiplication but not for division.

$$6 \times 5 = 30$$

5 × 6 = 30

$$27 \div 3 = 9$$

3 ÷ 27 does not equal 9

$$25 \div 5$$

5 ÷ 25 does not equal 5

$$4 \times 9 = 36$$

9 × 4 = 36

$$8 \times 7 = 56$$

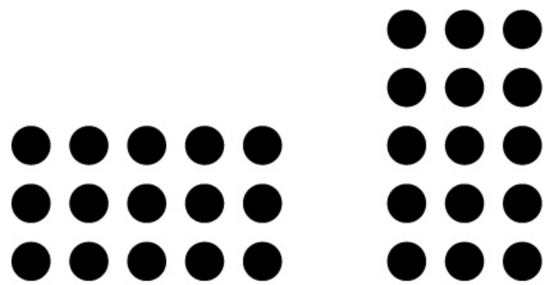
7 × 8 = 56

$$16 \div 2 = 8$$

2 ÷ 16 does not equal 8

- Ask students to explain or illustrate why reordering works for multiplication but not division. Some possible student responses are shown.

Multiplying two numbers, like 3 × 5, means creating a set of 3 rows with 5 dots in each row. Reordering the digits to 5 × 3 would reverse the numbers, creating a set of 5 rows with 3 dots in each row. The two sets have the same number of dots because turning one set on its side makes it perfectly match the other one.



3 × 5 5 × 3

Dividing two numbers, like 10 ÷ 2, means arranging 10 dots into 2 rows of equal dots. Reordering the digits to 2 ÷ 10 means arranging 2 dots into 10 rows of equal dots, which cannot be done without cutting up the dots.

Key Academic Terms:

product, sum, property of operations, multiplication expression, decompose, array model, area model, regroup, reorganize

Additional Resources:

- Activity: [Turn your array]
- Activity: [Valid equalities?]

6

Operations and Algebraic Thinking
Understand properties of multiplication and the relationship between multiplication and division. *Note: Students need not use formal terms for these properties.*
6. Use the relationship between multiplication and division to represent division as an equation with an unknown factor.

Guiding Questions with Connections to Mathematical Practices:

Why can a division problem be thought of as an unknown-factor problem?

M.P.3. Construct viable arguments and critique the reasoning of others. Understand that multiplication and division are related operations, and explain how the dividend in a division equation is the same as the product in a related multiplication equation. For example, $48 \div 6 = \square$ is equivalent to $\square \times 6 = 48$. Additionally, the two factors in a multiplication equation are the divisor and quotient of a related division equation.

- Ask students to identify the unknown number that makes a division equation and its related multiplication equation both true.
  - $\square \div 7 = 7$ and $7 \times 7 = \square$

    49
  - $16 \div \square = 2$ and $2 \times \square = 16$

    8
  - $56 \div 8 = \square$ and $\square \times 8 = 56$

    7

- Ask students to identify a multiplication equation that is related to a given division equation, and then use that to solve the division equation.
  - $12 \div 4 = \square$

    $\square \times 4 = 12$

    $\square = 3$
  - $54 \div 6 = \square$

    $\square \times 6 = 54$

    $\square = 9$

M.P.7. Look for and make use of structure. Demonstrate that a division problem in the form $b \div a$ is equivalent to "*a* times what number is equal to *b*?" or "How many groups of *a* are in *b*?" For example, $36 \div 9$ is equivalent to "9 times what number is equal to 36?" or $9 \times \square = 36$. Additionally, once the unknown factor is found, a total of two multiplication and two division equations can be formed by using the relationship between multiplication and division.

- Ask students to study a division problem in the form $b \div a$ and complete the equivalent unknown factor statement.
  - $24 \div 6$

    6 times ____ is equal to 24.

    4
  - $15 \div 3$

    There are 3 groups of ___ in 15.

    5
  - $35 \div 5$

    5 times ____ is equal to 35.

    7

- Ask students to identify if each statement is true or false.

There are 8 groups of 8 in 64.	true
There are 49 groups of 7 in 7.	false
54 times 9 is equal to 6.	false
6 times 5 is equal to 30.	true
Since $4 \times 4 = 16$, then $4 \div 16 = 4$.	false
Since $24 \div 3 = 8$, then $3 \times 8 = 24$.	true

- Ask students to write one division equation and one multiplication equation that are related to the given statement.
  - There are 9 groups of 8 in 72.

    $9 \times 8 = 72$

    $72 \div 9 = 8$
  - 7 times 6 is equal to 42.

    $7 \times 6 = 42$

    $42 \div 7 = 6$

Key Academic Terms:

product, quotient, multiplication, division, unknown factor, equivalent, times, related

Additional Resources:

- Activity: Division as an unknown factor
- Lessons: Grade 3 mathematics module 1, topic B, overview

7a

Operations and Algebraic Thinking
Multiply and divide within 100.
7. Use strategies based on properties and patterns of multiplication to demonstrate fluency with multiplication and division within 100. a. Fluently determine all products obtained by multiplying two one-digit numbers.

Guiding Questions with Connections to Mathematical Practices:

What does it mean to fluently multiply and divide within 100?

M.P.2. Reason abstractly and quantitatively. Use a combination of known facts, patterns, relationships, and other strategies to strategically, efficiently, and accurately multiply and divide within 100. For example, 4×9 can be found by doubling 9 to get 18, then doubling again to get 36 because $2 \times 2 \times 9 = 4 \times 9$. Another strategy is to think of 4×9 as 1 group of 4 less than 10 groups of 4. And 10 groups of 4 is 40, so 9 groups of 4 is $40 - 4 = 36$. Additionally, when trying to decide which strategy or combination of strategies to use when solving a multiplication or division problem, consider the numbers that are given and what is already known about the relationship between those numbers.

- Ask students to show or explain how to use a strategy to solve multiplication problems within 100.
  - 7×6

    42

    $7 \times 5 = 35$ and then add 1 more group of 7 to 35 to get $35 + 7 = 42$.
  - 15×5

    75

    Split 15 into 10 and 5, and then $(10 \times 5) + (5 \times 5) = 50 + 25 = 75$.

- 13×4

  52

  Double 13 to get 26, and then double 26 to get 52.

- Ask students to show or explain how to use a strategy to solve division problems within 100.
  - $48 \div 8$

    6

    If $6 \times 8 = 48$, then $48 \div 8$ must equal 6.

  - $65 \div 5$

    13

    $50 + 15 = 65$, so $(50 \div 5) + (15 \div 5)$ is the same as $65 \div 5$, so $10 + 3 = 13$.

  - $64 \div 4$

    16

    Divide 64 by 2 to get 32, and then divide 32 by 2 to get 16.

How does knowing one multiplication fact translate into knowing all the related multiplication and division facts?

M.P.7. Look for and make use of structure. Relate multiplication and division to identify all the multiplication and division facts that are related. For example, knowing that $3 \times 6 = 18$ also means knowing that $6 \times 3 = 18$, $18 \div 3 = 6$, and $18 \div 6 = 3$. Additionally, this combination of two multiplication equations and two division equations is known as related equations.

- Ask students to study one multiplication fact and then identify the related equations (one more multiplication fact and two division facts).
  - $9 \times 5 = 45$

    $5 \times 9 = 45$

    $45 \div 5 = 9$

    $45 \div 9 = 5$
  - $3 \times 8 = 24$

    $8 \times 3 = 24$

    $24 \div 8 = 3$

    $24 \div 3 = 8$

- Ask students to study one division fact and then identify the related equations (one more division fact and two multiplication facts).
  - $16 \div 2 = 8$

    $16 \div 8 = 2$

    $8 \times 2 = 16$

    $2 \times 8 = 16$
  - $50 \div 5 = 10$

    $50 \div 10 = 5$

    $10 \times 5 = 50$

    $5 \times 10 = 50$

Key Academic Terms:

fluently, product, dividend, fact, properties of operations

Additional Resources:

- Book: O'Connell, S., & SanGiovanni, J. (2014). *Mastering the basic math facts in multiplication and division*. Portsmouth, NH: Heinemann.
- Game: Demolition division
- Game: Pony pull division game

7b

Operations and Algebraic Thinking
Multiply and divide within 100.
7. Use strategies based on properties and patterns of multiplication to demonstrate fluency with multiplication and division within 100. b. State automatically all products of two one-digit numbers by the end of third grade.

Guiding Questions with Connections to Mathematical Practices:

How does fluency lead to automatically knowing products?

M.P.7. Looking for and making use of structure. Build fluency in multiplication and division to build automatic knowledge of products. For example, practice skip-counting by 2 and then move into skip-counting by 4. This leads to being able to quickly remember $8 \times 4 = 32$. Additionally, multiplying by 5 can be found by multiplying by 10 and then dividing by 2. For example, find the product of 5×7 by first finding $10 \times 7 = 70$ and then dividing 70 by 2 to find that 35 is the product of 5×7.

- Ask students to discuss how to use skip-counting by 2 to know the multiplication facts for 8. Help students realize that multiplying a multiple of 2 by 4 will get a multiple of 8. For example, 3×8 can be found by first knowing that $3 \times 2 = 6$ and then realizing that $6 \times 4 = 24$, so $3 \times 8 = 24$. For some students, the multiples of 6 may be easier to know automatically than the multiples of 8. Remind students that this strategy can be used for remembering the multiples of 6 as well. The multiples of 3 can be multiplied by 2 to find the multiples of 6. For example, to find 6×7, first find that 3×7 is 21. Then multiply 21 by 2 to find that $42 = 6 \times 7$.

Key Academic Terms:

fluently, product, dividend, fact, properties of operations

Additional Resources:

- Book: O'Connell, S., & SanGiovanni, J. (2014). *Mastering the basic math facts in multiplication and division*. Portsmouth, NH: Heinemann.
- Game: Demolition division
- Game: Pony pull division game

8

Operations and Algebraic Thinking
Solve problems involving the four operations and identify and explain patterns in arithmetic.
8. Determine and justify solutions for two-step word problems using the four operations and write an equation with a letter standing for the unknown quantity. Determine reasonableness of answers using number sense, context, mental computation, and estimation strategies including rounding.

Guiding Questions with Connections to Mathematical Practices:

How can a two-step word problem be modeled with an equation?

M.P.4. Model with mathematics. Identify important quantities from the context of a situation, and create an equation with a letter standing for the unknown quantity, known as a variable. For example, the situation "Ethan has 3 packages of markers with 4 markers in each pack. Jamal has 17 markers. How many more markers does Jamal have than Ethan?" can be modeled using the equation $3 \times 4 + m = 17$, where the variable m is how many more markers Jamal has than Ethan. Additionally, there is often more than one way in which an equation can be written to model a two-step word problem.

- Ask students to study a two-step word problem and given equations and to identify what the letter stands for in each problem.
  - Problem: Alex has 4 boxes of trains with 10 trains in each box. He puts the trains onto 5 different tracks. He puts the same number of trains on each track. How many trains are on each track?

    Two steps: $4 \times 10 = 40$

    $40 \div t = 5$

    The letter t stands for the number of trains on each track.

- Problem: Allie's teacher tells her to read for a total of 100 minutes within 5 days. On the first day, Allie reads for 12 minutes. Allie plans to read for the same number of minutes for each of the next 4 days. How many minutes should she read for on each of the next 4 days so that she gets to the total of 100 minutes?

  Two steps: $100 - 12 = 88$

  $4 \times m = 88$

  The letter m stands for the number of minutes Allie should read for on each of the remaining four days.

- Ask students to write an equation or equations to model each two-step word equation. Each equation will have a letter that stands for the unknown quantity.

  - Lila buys 3 muffins at the bakery and then goes home and bakes 12 muffins. After sharing some muffins with her friends, she has 4 muffins left. How many muffins did she give to her friends?

    $3 + 12 = 15$

    $15 - m = 4$

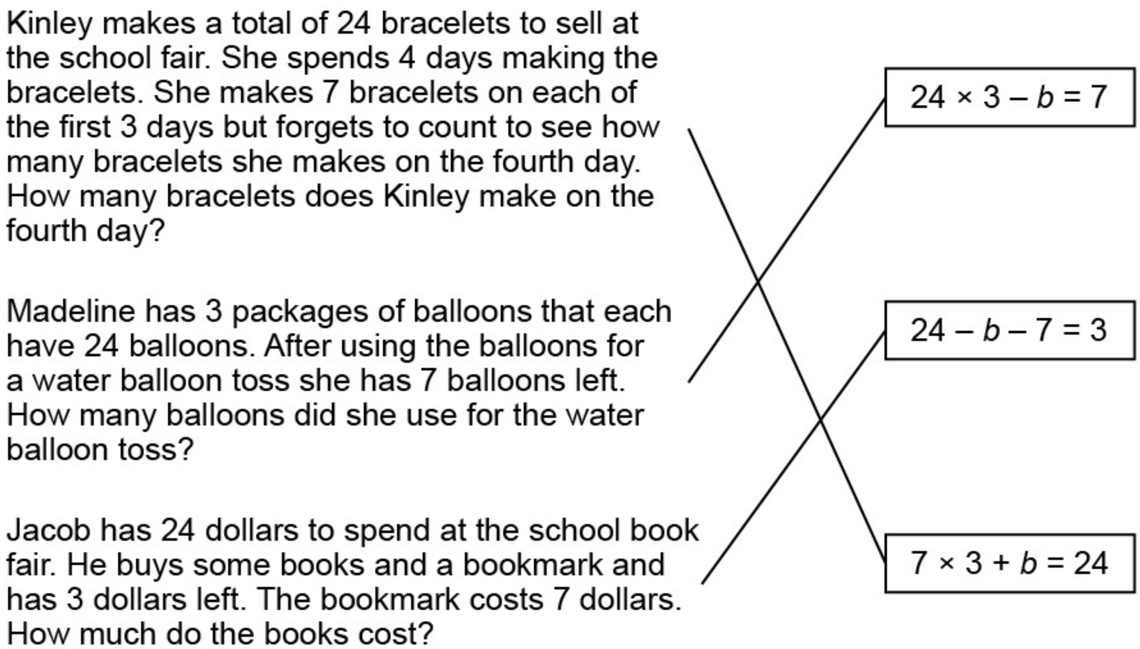
  - Nathan sells tickets for the school play. He sells 34 tickets on Monday and 23 tickets on Tuesday. The auditorium where the play will be performed can hold 112 people. How many more tickets can Nathan sell before the auditorium is full?

    $34 + 23 + t = 112$

    OR

    $112 - 34 - 23 = t$

- Ask students to match each two-step word problem with an equation that could be used to represent it.



How are expressions and equations with multiple operations solved?

M.P.7. Look for and make use of structure. Use the properties of operations and the context of the situation to solve two-step expressions and equations. For example, the situation "Marcus has 3 apples. He purchases an additional 2 bags of 8 apples each. How many apples does he have in all?" can be modeled using the expression $3 + 2 \times 8$ and evaluated by first multiplying 2 and 8 to get 16, then adding 3 for a sum of 19 apples.

- Ask students to evaluate expressions.
  - $(4 \times 6) + 7$

    31

  - $12 - (3 \times 2)$

    6

  - $(48 \div 6) + 3$

    11

- Ask students to write and solve a two-step equation that could be used to model a given situation.
    - Ryan scores 6 points in each of the first 3 quarters of his basketball game and then 9 points in the last quarter. How many points does he score in the game in all?

$$6 \times 3 + 9 = x$$

$$x = 27 \text{ points}$$

OR

$$6 \times 3 = 18$$

$$18 + 9 = 27 \text{ points}$$

    - Harper collects 46 seashells at the beach. She gives 6 of the shells to her little brother and then divides the rest of the shells equally into 5 buckets. How many shells are in each bucket?

$$46 - 6 = 40$$

$$40 \div 5 = x$$

$$x = 8 \text{ shells}$$

How can the reasonableness of an answer be checked using mental computation and estimation?

M.P.3. Construct viable arguments and critique the reasoning of others. Detect possible errors by using estimation and the context of a word problem. For example, the situation “Rachel needs 32 cookies to share with her class, and she has already baked 24 cookies. How many more cookies will Rachel need to bake?” will have an answer close to 10 because 32 is close to 30 and 24 is close to 20 and 30 – 20 = 10. Additionally, estimation is a fast and easy way to help determine if the operations that were selected to solve a two-step word problem were correct, if the order of operations was performed correctly, and/or to see if any computation errors were potentially made while solving the problem.

- Ask students to use estimation to show if the given answer to a problem is most likely correct or incorrect. If students find the answer to most likely be incorrect, ask students to solve the problem to find the actual correct answer.
    - Word Problem: It takes Sophie 19 minutes to ride her bike to school. She rode her bike to school 4 days last week. On the fifth day she was driven to school and that took 5 minutes. What is the total number of minutes it took Sophie to get to school last week?

      Answer: 81 minutes

      20 × 4 + 5 = 85; the answer is most likely correct since 81 is close to 85.

    - Word Problem: A pet store has 9 tanks that each have 4 fish in them. The pet store then gets 24 more fish to add to the tanks. How many total fish does the pet store have now?

      Answer: 37 fish

      10 × 4 + 25 = 65 fish; the answer is most likely incorrect since 37 is not close to 65. The actual answer is 60 since 9 × 4 + 24 = 60.

- Ask students to match the word problem to the estimation equation that would be most helpful to check the reasonableness of the actual answer.

Emma eats 3 meals and 2 snacks a day. How many total times does she eat in 28 days?

Holden has a total of 28 baseball cards. He gives 2 to his little brother and 3 to his neighbor. How many baseball cards does he have left?

Eli has 28 markers that he divides equally into a bucket and a pencil box. He then takes 3 markers out of the bucket. How many markers are left in the bucket?

30 ÷ 2 – 3 = 12

30 – (2 + 3) = 25

(3 + 2) × 30 = 150

Key Academic Terms:

unknown quantity, mental computation, estimation, variable, two-step problem, equation, expression, rounding, reasonableness

Additional Resources:

- Video: Solving two-step word problems
- Video: Two-step word problems
- Lesson: The class trip
- Lesson: The stamp collection

9

Operations and Algebraic Thinking
Solve problems involving the four operations and identify and explain patterns in arithmetic.
9. Recognize and explain arithmetic patterns using properties of operations.

Guiding Questions with Connections to Mathematical Practices:

How can an arithmetic pattern be identified?

M.P.3. Construct viable arguments and critique the reasoning of others. Illustrate that when consecutive terms always differ by the same amount, an arithmetic pattern is formed. For example, in the pattern 4, 7, 10, 13, . . ., each term in the pattern differs from the previous term by 3. A visual pattern can also be found in the multiplication table, for example, by shading all the even numbers. The pattern results in all the numbers in the second, fourth, sixth, eighth, and tenth columns being shaded. Additionally, a checkerboard pattern of even and odd numbers can be found in an addition table.

- Ask students to identify which number would come next in an arithmetic pattern.
    - 3, 7, 11, 15, ____

      19
    - 6, 12, 18, 24, ____

      30

- Ask students to identify if a given pattern is an arithmetic pattern or not.
  - 4, 8, 10, 12, 14

    not an arithmetic pattern
  - 0, 2, 4, 6, 8

    arithmetic pattern
  - 1, 3, 6, 9, 12

    not an arithmetic pattern
- Ask students to study an addition table and a multiplication table and to identify any patterns they observe in the tables. Then, lead a class discussion to write some general rules about addition and multiplication. Some rules could include: an even number plus an even number is always even, an odd number plus an odd number is always even, an even number plus an odd number is always odd, a number multiplied by an even number is always even, a number multiplied by 1 is always the number itself, a number multiplied by 0 is always 0, etc.

How can an arithmetic pattern be described?

M.P.1. Make sense of problems and persevere in solving them. Demonstrate that an arithmetic pattern can be described by the starting value and an addition or subtraction rule. The pattern can also be connected to skip-counting and multiplication. For example, the pattern 5, 9, 13, 17, . . . can be described as "start with 5, then add 4 each time," and the pattern 6, 12, 18, 24, . . . is the same pattern that results when starting at 6 and skip-counting by 6. Additionally, observe that to find the second number in the pattern the rule needs to be applied once, to find the third number in the pattern the rule needs to be applied twice, etc. Further, other types of arithmetic patterns can be found when using an addition or multiplication table—a visual pattern may be described.

- Ask students to name the first 5 numbers in a pattern when given a start number and a rule about the pattern.
  - Start with 7, then add 2 each time.

    7, 9, 11, 13, 15
  - Start with 0, then add 5 each time.

    0, 5, 10, 15, 20
  - Start at 35, then subtract 7 each time.

    35, 28, 21, 14, 7
- Ask students to study a pattern and then identify the start number and the rule that was used to create that pattern.
  - 1, 4, 7, 10, 13

    Start at 1, then add 3 each time, or start at 1, then skip-count by 3.
  - 4, 8, 12, 16, 20

    Start at 4, then add 4 each time, or start at 4, then skip-count by 4.
  - 14, 12, 10, 8, 6

    Start at 14, then subtract 2 each time, or start at 14, then skip-count backward by 2.

Key Academic Terms:

arithmetic pattern, starting value, addition table, multiplication table, consecutive, term, decompose

Additional Resources:

- Lesson: Patterns in multiplication
- Activity: Addition patterns

10

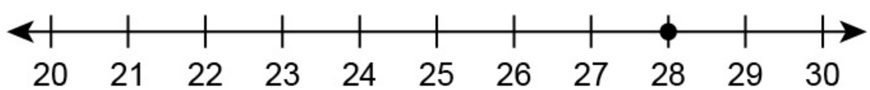
Operations with Numbers: Base Ten
Use place value understanding and properties of operations to perform multi-digit arithmetic.
10. Identify the nearest 10 or 100 when rounding whole numbers, using place value understanding.

Guiding Questions with Connections to Mathematical Practices:

How are numbers rounded to the nearest 10 or 100?

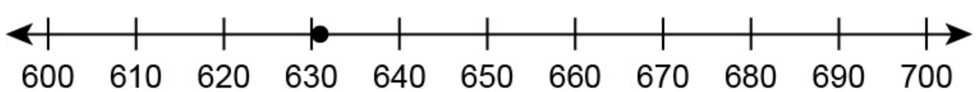
M.P.5. Use appropriate tools strategically. Demonstrate on a number line how to round a given number to the nearest 10 or 100. For example, when rounding 56 to the nearest 10, find the location of 56 on the number line and determine the closest 10 is 60. Additionally, demonstrate that the place being rounded to determines which values to consider on the number line; when rounding to 10s, nearby multiples of 10 are considered, and when rounding to 100s, nearby multiples of 100 are considered.

- Ask students to round a two-digit number to the nearest 10 using a number line. For example, given the number 28, consider the part of the number line containing 28 and the multiples of 10 just before and just after 28, in this case 20 and 30. The relevant part of the number line with a point plotted at 28 is shown.

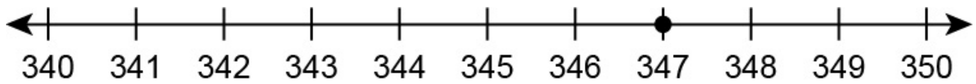
  Because the distance from 28 to 30 is only 2 units and the distance from 28 to 20 is 8 units, the value of 28 when rounded to the nearest 10 is 30.

- Ask students to round a three-digit number to the nearest 100 using a number line. For example, given the number 631, consider the part of the number line containing 631 and the multiples of 100 just before and after 631, in this case 600 and 700. The relevant part of the number line with a point plotted at 631 is shown.

  Because the distance from 631 to 600 is slightly more than 3 tens (30 units) and the distance from 631 to 700 is slightly less than 7 tens (70 units), the value of 631 when rounded to the nearest 100 is 600.

- Ask students to round a three-digit number to the nearest 10 using a number line. For example, given the number 347, consider the part of the number line containing 347 and the multiples of 10 (not 100) just before and after 347, in this case 340 and 350. The relevant part of the number line with a point plotted at 347 is shown.

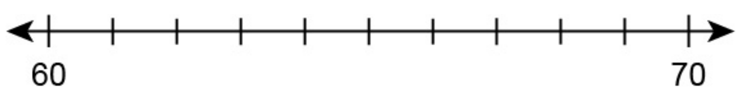


Because 347 is being rounded to the nearest 10, only consider the distance on the number line from 347 to the two nearest multiples of 10. Because the distance from 347 to 350 is 3 units and the distance from 347 to 340 is 7 units, the value of 347 when rounded to the nearest 10 is 350.

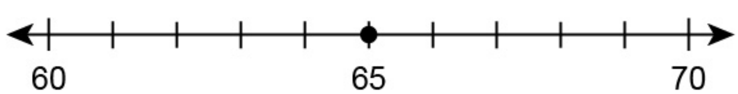
What makes "5" significant when rounding to the nearest 10?

M.P.7. Look for and make use of structure. Identify that 5 is significant because it represents the halfway point between two 10s on a number line. For example, on a number line from 10 to 20, the interval between 10 and 20 is divided into ten equally sized sections that are marked with the numbers 11 through 19. The number 15 is the same distance from 10 as it is from 20 on the number line, so it represents the halfway point between 10 and 20. To the left of 15, all the values are closer to 10, and to the right of 15, all of the values are closer to 20. The value of 15 will round up to 20 because half of the values round to 10 and half of the values round to 20. Additionally, know that a 5 in the ones place is only significant when rounding to the nearest 10, even if the number being rounded extends to the hundreds place or beyond.

- Ask students to identify the halfway point between two consecutive 10s on a number line. For example, give students the following number line.



Because there are 10 intervals on the number line between 60 and 70, the halfway point between 60 and 70 is the point that has 5 intervals to its left and 5 intervals to its right. That point is 65 and can be located by counting up by 1s from 60 or counting down by 1s from 70. The location of 65 is shown on the number line.

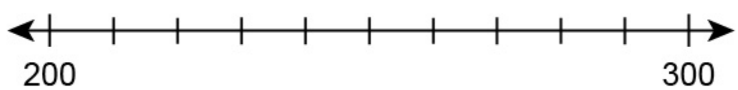


- Ask students to round numbers with a 5 in the ones place to the nearest 10. For example, to round 35 to the nearest 10, confirm the ones place as the relevant place. Also, know that 35 is between 30 and 40. Because there is a 5 in the ones place, 35 is rounded up to the nearest 10, which is 40. As an additional example, to round 285 to the nearest 10, again confirm the ones place as the relevant place (because the rounding is to the tens place, even though the number 285 extends to the hundreds place). Because there is a 5 in the ones place, 285 is rounded up to the nearest 10, which is 290.

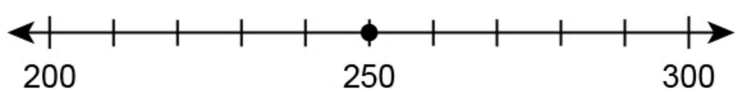
What makes “50” significant when rounding to the nearest 100?

M.P.7. Look for and make use of structure. Identify that 50 is significant because it represents the halfway point between two 100s on a number line. For example, on a number line from 300 to 400, the interval is divided into ten equally sized sections that are marked with the numbers 310, 320, 330, 340, 350, 360, 370, 380, and 390. The number 350 is the same distance from 300 as it is from 400 on the number line, so it represents the halfway point between 300 and 400. To the left of 350 all the values are closer to 300, and to the right of 350 all the values are closer to 400. The value of 350 will round up to 400 because half the values round to 300 and half the values round to 400. Additionally, know that a 5 in the tens place is only significant when rounding to the nearest 100.

- Ask students to identify the halfway point between two consecutive 100s on a number line. For example, give students the following number line.



Because there are 10 intervals of 10 units in length between 200 and 300, the halfway point between 200 and 300 is the point that has 5 intervals to its left and 5 intervals to its right. That point is 250 and can be located by counting up by 10s from 200 or counting down by 10s from 300. The location of 250 is shown on the number line.



- Ask students to round numbers with a 5 in the tens place to the nearest 100. For example, to round 752 to the nearest 100, confirm the tens place as the relevant place. Also, know that 752 is between 700 and 800. Because there is a 5 in the tens place, 752 is rounded up to the nearest 100, which is 800.

Key Academic Terms:

place value, round, nearest 10, nearest 100, multi-digit, halfway point

Additional Resources:

- Activity: Roll it! rounding game
- Activity: Rounding to 50 or 500

11

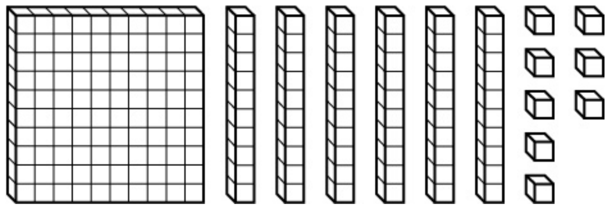
Operations with Numbers: Base Ten
Use place value understanding and properties of operations to perform multi-digit arithmetic.
11. Use various strategies to add and subtract fluently within 1000.

Guiding Questions with Connections to Mathematical Practices:

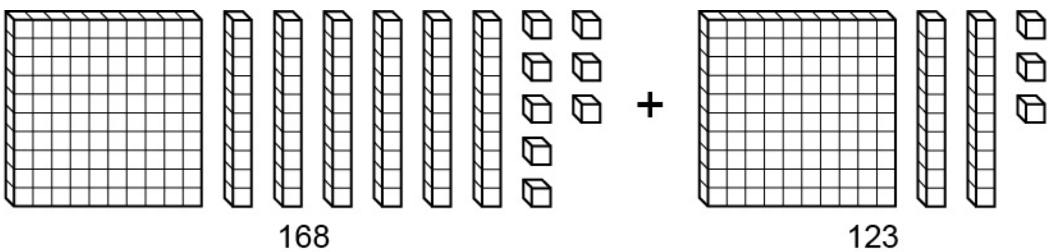
How can the sum of two numbers be found?

M.P.5. Use appropriate tools strategically. Demonstrate how to add two numbers using a strategy involving paper and pencil, tools such as base-ten blocks, place value, properties of operations, and/or the relationship between addition and subtraction. This prepares students to learn the standard algorithm, which is introduced in grade 4. For example, add 199 + 57 by demonstrating it is the same as 200 + 57 – 1. Additionally, demonstrate regrouping using base-ten blocks.

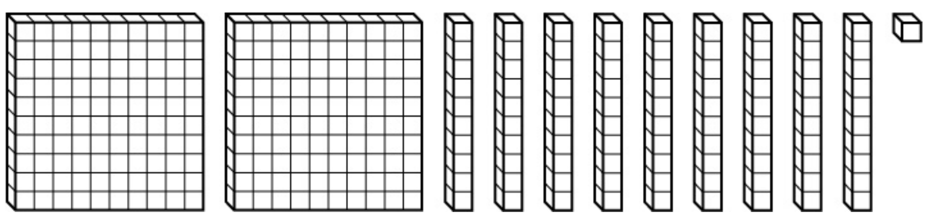
- Ask students to use base-ten blocks to show addition. For example, to add 168 + 123, first represent 168 with base-ten blocks, as shown.



Then, add 1 hundred-block, 2 ten-blocks, and 3 one-blocks to the figure to represent adding 123 to 168.



Because there are 11 one-blocks, replace 10 of them with a single ten-block, leaving a single one-block. Then, count the base-ten blocks to determine the sum.



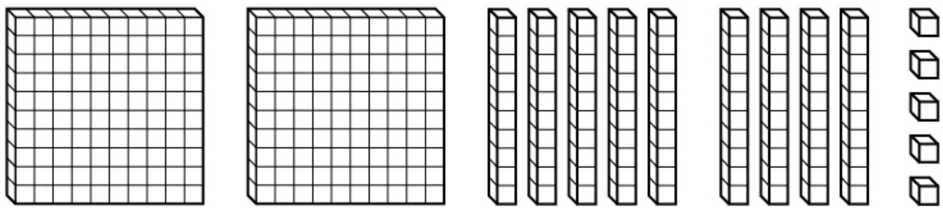
Because there are 2 hundred-blocks, 9 ten-blocks, and 1 one-block, $168 + 123 = 291$.

- Ask students to add two numbers by rewriting one of the numbers as a nearby number combined with adding or subtracting a small number. For example, add $349 + 26$ by changing 349 to $350 – 1$ and rewriting the addition problem as $350 + 26 – 1$. Since $26 – 1 = 25$, the value of $349 + 26$ is 375. As an additional example, add $203 + 77$ by changing 203 to $200 + 3$ and rewriting the addition problem as $200 + 77 + 3$. Since $77 + 3 = 80$, the value of $203 + 77$ is 280.

How can the difference of two numbers be found?

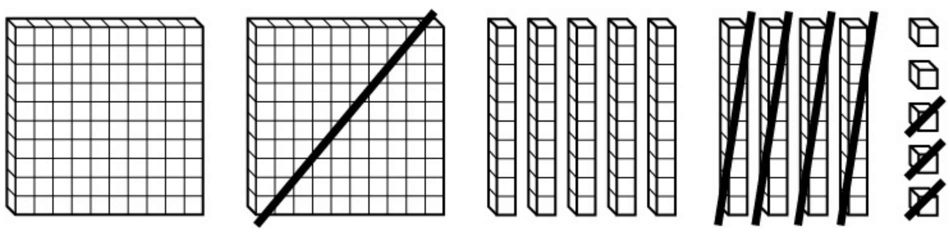
M.P.5. Use appropriate tools strategically. Demonstrate how to subtract two numbers using a strategy involving paper and pencil, tools such as base-ten blocks, place value, properties of operations, and/or the relationship between addition and subtraction. For example, subtract 341 – 236 by thinking of it as 236 + ___ = 341 and adding up on a number line or by taking 236 away from 341. Additionally, subtract 374 – 77 by rewriting the problem as 374 – 74 – 3 and performing two separate subtractions.

- Ask students to use base-ten blocks to show subtraction. For example, to subtract 295 – 143, first represent 295 with base-ten blocks, as shown.

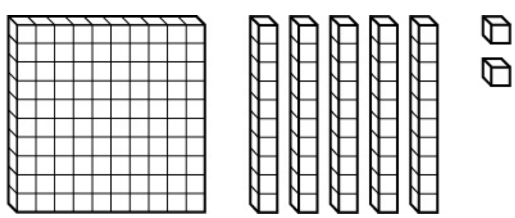
  

  295

  Then, remove the blocks representing 143, which are 1 hundred-block, 4 ten-blocks, and 3 one-blocks, leaving the blocks shown.

  Finally, count the remaining base-ten blocks. Since there are 1 hundred, 5 tens, and 2 ones, the value of 295 – 143 = 152.

- Ask students to rewrite subtraction problems as addition problems with a missing addend. For example, 784 – 167 can be rewritten as 167 + ___ = 784. Students should discuss why a given subtraction problem cannot be rewritten as, for example, 784 + ___ = 167.

- Ask students to subtract two numbers by rewriting one of the numbers as a nearby number combined with adding or subtracting a small number. For example, subtract $953 - 99$ by changing 99 to $100 - 1$ and rewriting the subtraction problem as $953 - 100 + 1$. Since $953 - 100 = 853$ and $853 + 1 = 854$, the value of $953 - 99 = 854$.

Key Academic Terms:

sum, difference, base-ten blocks, place value, multi-digit, properties of operations, algorithm

Additional Resources:

- Lesson: Addition—expanded form
- Lesson: Mental math with place value
- Lesson: Subtraction—expanded form

12

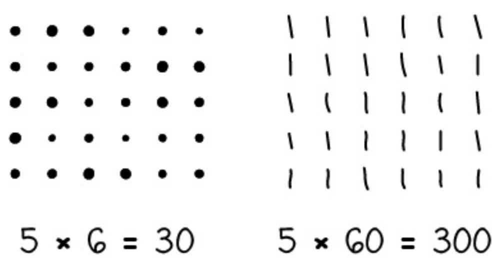
Operations with Numbers: Base Ten
Use place value understanding and properties of operations to perform multi-digit arithmetic.
12. Use concrete materials and pictorial models based on place value and properties of operations to find the product of a one-digit whole number by a multiple of ten (from 10 to 90).

Guiding Questions with Connections to Mathematical Practices:

How does multiplying a one-digit number by a multiple of 10 connect to multiplying two one-digit numbers?

M.P.8. Look for and express regularity in repeated reasoning. Extend multiplication of two one-digit numbers to multiplying a one-digit number by a multiple of 10 using understanding of place value and properties of operations. For example, know that because $6 \times 4 = 24$, 6×40 can be thought of as 6×4 tens, which is 24 tens or 240. Or, 4×70 is the same as $4 \times (7 \times 10)$ which equals $(4 \times 7) \times 10$ or $28 \times 10 = 280$. Additionally, illustrate that multiplying a one-digit number by a multiple of ten increases the place value of the digits in the product by one place when compared to multiplying two one-digit numbers.

- Ask students to describe the number of tens present in the product of multiplication problems involving a multiple of 10 by using manipulatives or drawings. For example, illustrate that 5×60 is equivalent to 5×6 tens. Since $5 \times 6 = 30$, the product 5×60 is equivalent to 30 tens, or 300. The figure demonstrates this, with dots being used for ones and "sticks" being used to represent tens.



Additionally, students should verify that 20×7 is equivalent to 2×7 tens. Since $2 \times 7 = 14$, the product 20×7 is equivalent to 14 tens, or 140. This can be shown using base-ten blocks.

- Ask students to rewrite multiplication problems involving a 10 using properties of multiplication. For example, the expression 5×90 can be rewritten as $5 \times (9 \times 10)$, since $9 \times 10 = 90$. This expression can be rewritten as $(5 \times 9) \times 10$, since numbers can be grouped and multiplied in any order. Since $5 \times 9 = 45$, the expression is equivalent to 45×10, which is equal to 450.

- Ask students to find the new place values of the digits in the product of a one-digit number and a multiple of 10 by first finding the product of two one-digit numbers. For example, since $3 \times 7 = 21$, the problem $(3 \times 7) \times 10$ means that the 2 tens and 1 one in the product 3×7 each get multiplied by 10, so the final product will have 2 hundreds, 1 ten, and no ones, giving a final product of 210. Illustrate that this process also gives two closely related multiplication facts. In this case, $30 \times 7 = 210$ and $3 \times 70 = 210$. Additionally, since $9 \times 2 = 18$, the product $(9 \times 2) \times 10 = 180$. This also gives the two multiplication facts $9 \times 20 = 180$ and $90 \times 2 = 180$.

Key Academic Terms:

multiply, one-digit, multiple of 10, place value, properties of operations

Additional Resources:

- Video: Multiplying by multiples of 10
- Lesson: Multiplying with multiples of 10
- Game: Multiplying by multiples of 10

13

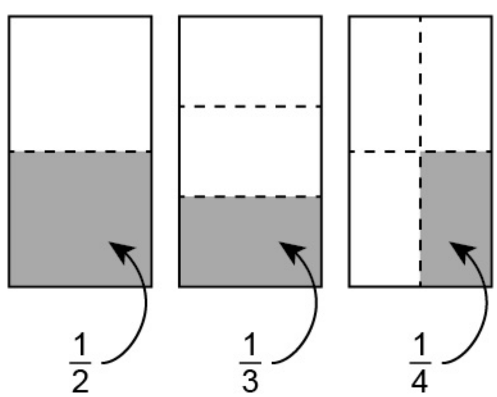
Operations with Numbers: Fractions
Develop understanding of fractions as numbers. *Denominators are limited to 2, 3, 4, 6, and 8.*
13. Demonstrate that a unit fraction represents one part of an area model or length model of a whole that has been equally partitioned; explain that a numerator greater than one indicates the number of unit pieces represented by the fraction.

Guiding Questions with Connections to Mathematical Practices:

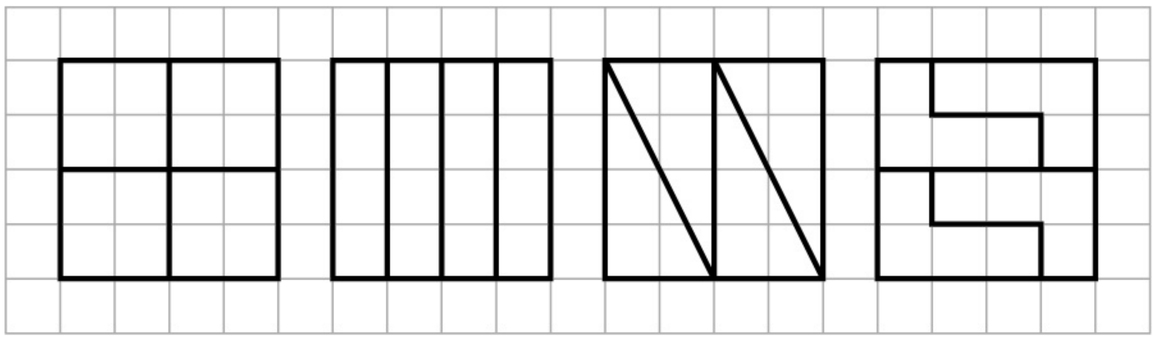
How can fractions describe the partitions of a shape?

M.P.7. Look for and make use of structure. Connect a partitioned shape to fractions of a whole, with the denominator being the total number of parts with equal areas. For example, when a rectangle is partitioned into 8 equally sized but differently shaped quadrilaterals, each will represent $\frac{1}{8}$ of the area of the whole rectangle. Additionally, the fraction represented is dependent only on the number of equally sized parts, not on the method used to partition the shape.

- Ask students to use a unit fraction to express the area of each part as a unit fraction of the whole, given a shape that has been partitioned into equally sized parts. For example, when a shape is divided into 2 equal parts, 1 part is called “one-half”; when a shape is divided into 3 equal parts, 1 part is called “one-third”; and when a shape is divided into 4 equal parts, 1 part is called “one-fourth.”



- Ask students to explain why two different methods of partitioning the same shape into the same number of equally sized parts result in pieces that are the same unit fraction of the whole. For example, give each student a square and ask the students to partition the square into four equally sized parts. Some possible methods are shown.

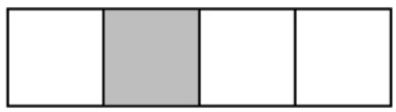


Compare the different methods that students used to partition their squares and observe that though students may have partitioned their squares in different ways, each piece has an area that is $\frac{1}{4}$ of the area of the square because regardless of how the square was partitioned, each piece is one of four equal pieces that make up the entire square.

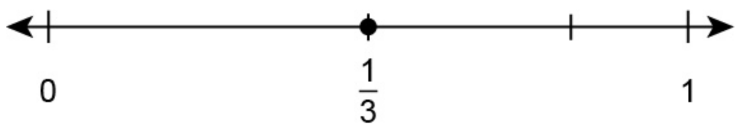
What do the numerator and denominator of a fraction represent?

M.P.2. Reason abstractly and quantitatively. Explain that the denominator of a fraction shows how many equally sized parts a whole is partitioned into, and that the numerator of a fraction shows the number of parts. For example, the fraction $\frac{3}{4}$ refers to 3 parts of a whole that was partitioned into 4 equal parts. Additionally, read fractions using fraction language: $\frac{2}{2}$ is read as “two halves” rather than “two twos,” “two over two,” or “two out of two.” Further, model fractions that are less than 1, equal to 1, and greater than 1 with a variety of hands-on materials.

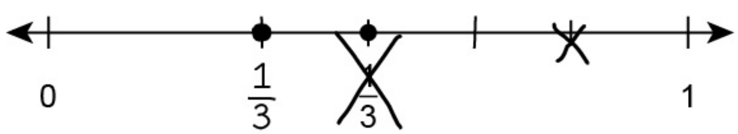
- Ask students to identify, define, and write the numerator and denominator of fractions represented using fraction notation, area models, manipulatives, and number lines. Identify that the number of parts of the whole is the denominator, and the number of the identified parts of the whole is the numerator. The area model below shows 1 shaded square in a rectangle partitioned into 4 equally sized squares, so students write the fraction $\frac{1}{4}$ and explain that there are 4 equally sized parts in the whole that represent the denominator and the 1 shaded part represents the numerator.



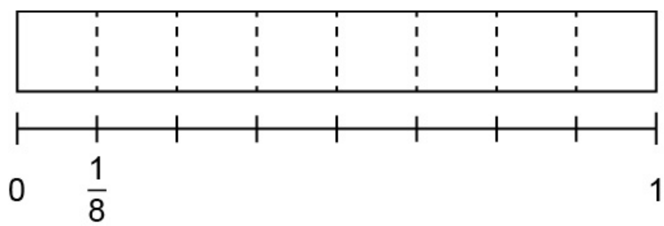
- Ask students to represent fraction values using visual models (e.g., fraction circles, fraction bars, pattern blocks, paper strips, connecting cubes). Implement student-created manipulatives to explore fractions, choosing which manipulatives to use when representing and solving problems. Make connections between and among the representations to make meaning of the fractions. Also, use models where parts are unequal in size to find non-examples. A number line like the one shown is divided into 3 unequal parts. Students should identify that the fraction $\frac{1}{3}$ does not accurately describe the value indicated on the number line.



Have students cross out the incorrect placement of $\frac{1}{3}$. Then, have students mark and label the correct placement, as shown.

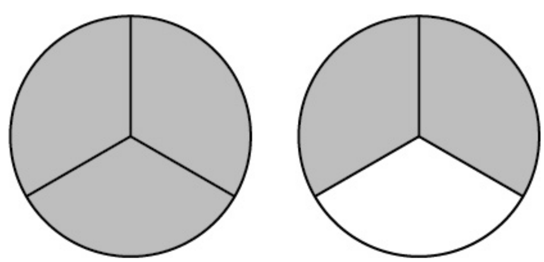


- Ask students to partition a variety of area models (including diagrams and paper strips) and number lines into 2, 3, 4, 6, or 8 parts. Connect the concept of the folded paper strips to the number line by identifying the whole on the number line based on the whole of the paper strip. The diagram shows a paper strip partitioned into 8 equal parts with a number line that directly aligns with the partitions.



Explain that fractions are one number, similar to whole numbers, and that no matter the number of digits in the numerator or denominator, each fraction has a single value. Just as 198 has 3 digits and represents one number with a single value, the fraction $\frac{1}{6}$ has two digits and represents a number with a single value.

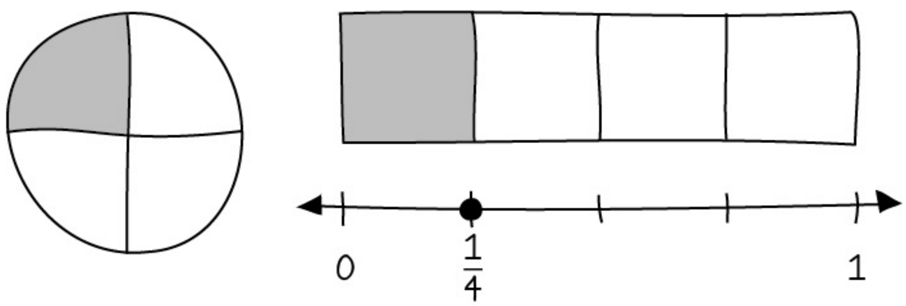
- Ask students to describe fractions verbally and in word form, and in terms of the unit fraction. Fractions should use the denominators of 2, 3, 4, 6, and 8, and may be less than, equal to, or greater than 1. The diagram shown depicts the fraction $\frac{5}{3}$, so students should respond "five-thirds," because it is equivalent to five one-thirds.



How can the unit fraction be determined from a visual representation?

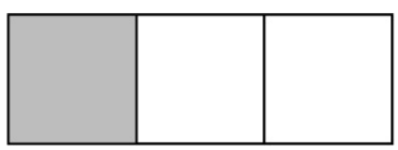
M.P.2. Reason abstractly and quantitatively. Explain that when a whole is divided into several equally sized parts, each of those parts represents 1 part of the whole that is written with 1 as the numerator and the number of equal parts as the denominator. For example, if a circle is partitioned into 8 equal pieces, each piece represents $\frac{1}{8}$ of the circle. Additionally, if a number line that begins with 0 and ends at 2 has tick marks at 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2, determine that the whole is 1, and that each part is $\frac{1}{2}$.

- Ask students to create, identify, and describe unit fractions using visual representations and models. For example, given the fraction $\frac{1}{4}$, students draw a model that accurately depicts $\frac{1}{4}$.



  Conversely, students identify the unit fraction, given a model.

- Ask students to write unit fractions in number and word form. Write "$\frac{1}{3}$" and "one-third" to describe the model shown.

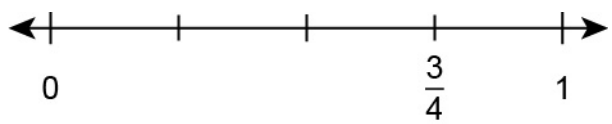


- Ask students to speculate about the connection between the size of a part of a whole (numerator) and the total number of parts of a whole (denominator). For example, make multiple comparisons between halves, thirds, fourths, sixths, and eighths using visual fraction representations where students can make the connection that the larger the denominator, the smaller the part.

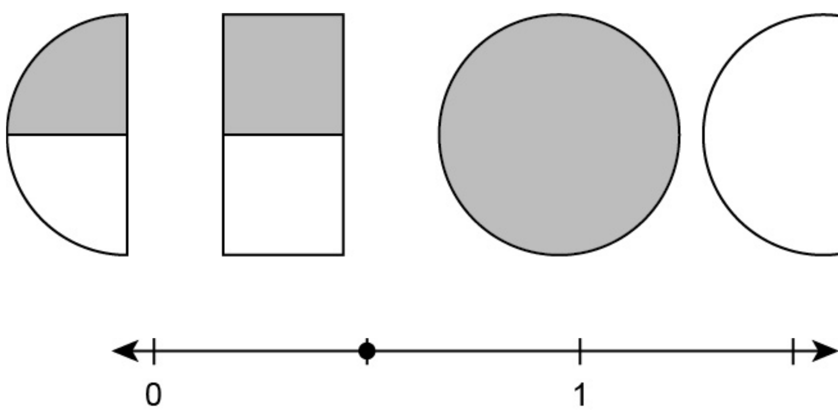
How can the whole of a fraction be determined?

M.P.6. Attend to precision. Determine the whole in order to determine the unit fraction. For example, given 2 circles with each partitioned into 4 equal parts and 7 parts shaded, it should be determined if the whole is one circle or two circles. If the whole is one circle, the unit fraction is $\frac{1}{4}$, so the fraction represented is $\frac{7}{4}$. If the whole is two circles, the unit fraction is $\frac{1}{8}$, so the fraction represented is $\frac{7}{8}$. Additionally, if given a number line, the whole is always 1.

- Ask students to identify the whole given in a variety of fraction models, including area models and number lines. For example, given the number line shown, identify that the whole is 1 and that each tick mark represents $\frac{1}{4}$ of that whole.



- Ask students to represent a fraction using a variety of different wholes. For example, students might represent $\frac{1}{2}$ with the following four drawings: a semicircle, a rectangle, a set of two circles, and a number line.



Key Academic Terms:

fraction, numerator, denominator, divide, part, whole, unit fraction, partition, equally sized parts, shape, decompose

Additional Resources:

- Book: Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: fractions & decimals*. Portsmouth, NH: Heinemann.
- Video: The coolest hands-on fraction activity ever!
- Video: The best way to learn fractions!

14a

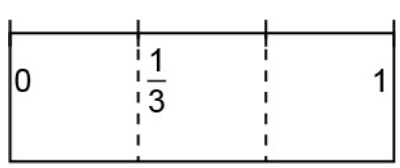
Operations with Numbers: Fractions
Develop understanding of fractions as numbers. *Denominators are limited to 2, 3, 4, 6, and 8.*
14. Interpret a fraction as a number on the number line; locate or represent fractions on a number line diagram. a. Represent a unit fraction ($\frac{1}{b}$) on a number line by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts as specified by the denominator.

Guiding Questions with Connections to Mathematical Practices:

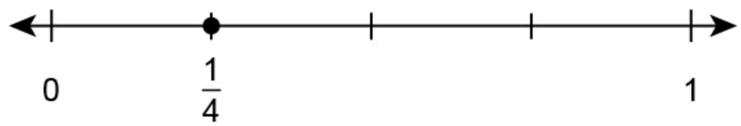
How is the location of a unit fraction on a number line determined?

M.P.5. Use appropriate tools strategically. Demonstrate how to partition and designate a whole number to represent a unit fraction given a blank number line. For example, given the fraction $\frac{1}{4}$, label a number line with 0 and 1, partition the line into 4 equal intervals by drawing 3 tick marks, and explain that $\frac{1}{4}$ is located at the first mark when counting by fourths from 0. Additionally, given the fraction $\frac{1}{8}$, fold a number line into eighths, unfold, mark the folds with tick marks, and explain that each part is $\frac{1}{8}$.

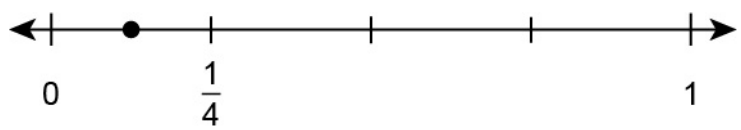
- Ask students to use paper strips to construct number lines and rulers in 2, 3, 4, 6, and 8 parts. In the diagram, students use the fold marks to determine the denominator of the fraction as 3 and note that each part is $\frac{1}{3}$ of the strip.



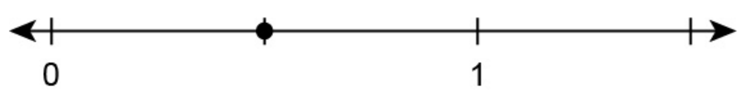
- Ask students to partition number lines by hand, using approximate locations. Students will label the number line with a minimum of two points to emphasize the unit. For example, students draw a number line marked with 0, 1, and $\frac{1}{4}$, and draw tick marks for the other $\frac{1}{4}$ sections.



- Ask students to estimate unit fractions on a number line. Given a number line with two points labeled, students determine the location of a fraction on that number line. Students will most likely have to partition the number line in order to best estimate values. Given the diagram, students estimate where $\frac{1}{8}$ is located on that number line.



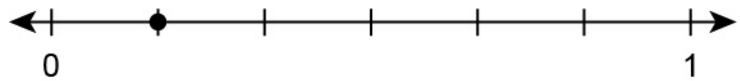
- Ask students to also identify unit fractions on number lines that are partitioned beyond 1, as shown in the diagram below, to make sense of the unit. For example, students plot the point and identify it as $\frac{1}{2}$.



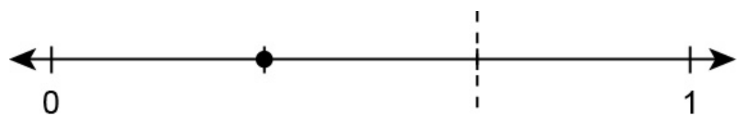
How is the value of a unit fraction on a number line determined?

M.P.2. Reason abstractly and quantitatively. Determine the value of a unit fraction on a number line given the point that represents the unit fraction on the number line and the whole. For example, if a number line with 0 and 1 marked is partitioned into eight equal parts and the point is located at the first tick mark after 0, the fraction represents $\frac{1}{8}$. Additionally, if a number line with 0 and 2 marked is partitioned into 4 equal parts, and the point is located at the first tick mark after 0, the fraction represents $\frac{1}{2}$.

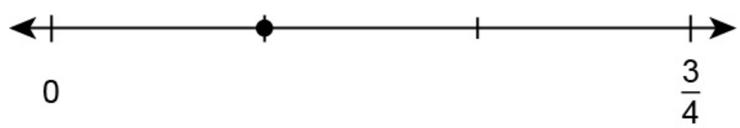
- Ask students to identify and label unit fractions with the denominators of 2, 3, 4, 6, and 8 on number lines. Students determine there are 6 sections from 0 to 1 in the diagram, so the denominator is 6, and the first tick mark represents 1 part, so the fraction is $\frac{1}{6}$.



- Ask students to determine the value of a unit fraction on a number line where the whole or denominator is not readily apparent. Students may need to add more partitions to approximate the fraction $\frac{1}{3}$, even though only 0, 1, and $\frac{1}{3}$ have tick marks placed.



Students will also identify the whole to determine the value of the fraction when the 1 is not labeled. In the diagram, students could extend the number line to 1 and add partitions to determine the value of the point is $\frac{1}{4}$.



Key Academic Terms:

number line, partition, tick mark, fraction, numerator, denominator, divide, equal parts, whole, unit fraction, equal intervals, value

Additional Resources:

- Activity: Fractions as numbers on a number line
- Game: Fractions on the number line

14b

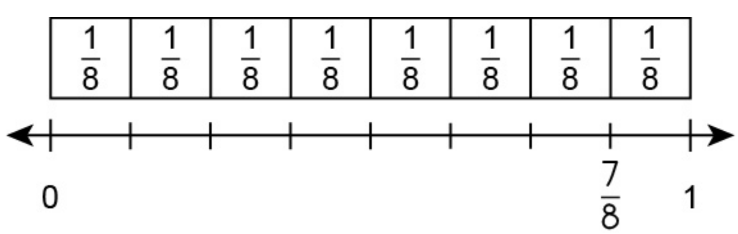
Operations with Numbers: Fractions
Develop understanding of fractions as numbers. *Denominators are limited to 2, 3, 4, 6, and 8.*
14. Interpret a fraction as a number on the number line; locate or represent fractions on a number line diagram. b. Represent a fraction ($\frac{a}{b}$) on a number line by marking off *a* lengths of size ($\frac{1}{b}$) from zero.

Guiding Questions with Connections to Mathematical Practices:

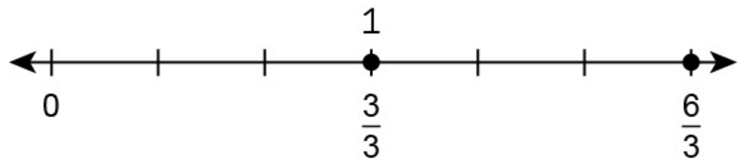
How is the location of any fraction on a number line determined?

M.P.5. Use appropriate tools strategically. Demonstrate how to partition and designate the whole to represent a fraction, given a blank number line. For example, given the fraction $\frac{2}{4}$, label a number line with 0 and 1, partition the line into four equal parts by drawing three tick marks, and explain that $\frac{2}{4}$ is located at the second mark when counting by fourths from 0. Additionally, create $\frac{1}{4}$-sized sections and copy them 5 times to show the fraction $\frac{5}{4}$.

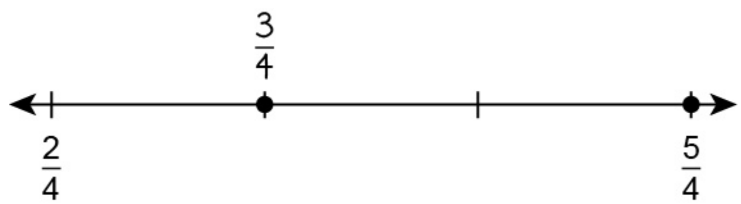
- Ask students to partition blank number lines and place fractions at the appropriate location, using grid paper to assist with even partitioning. For the fraction $\frac{7}{8}$, students partition a blank number line into eighths by copying a $\frac{1}{8}$ section 8 times, marking each $\frac{1}{8}$ section with a tick mark, then marking the end of the seventh section with a point labeled $\frac{7}{8}$.



- Ask students to place fractions at locations on partitioned number lines. Given a number line partitioned into thirds with only 0 labeled, place the fraction $\frac{6}{3}$. Students designate the whole to ensure the fraction is appropriately placed.



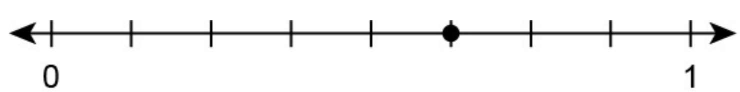
Students also place fractions on number lines that do not begin at 0. Students place the fraction $\frac{3}{4}$ on the number line by finding the unit and the denominator.



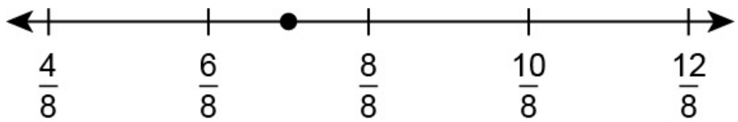
How is the value of any fraction on a number line determined?

M.P.2. Reason abstractly and quantitatively. Determine the value of a fraction on a number line given a point on a number line and the whole. For example, a number line is divided into eight equal intervals between 0 and 1 and another eight equal intervals between 1 and 2. If a point is located at the tenth tick mark after 0, count 10 units from 0 and state that the fraction represents $\frac{10}{8}$. Additionally, demonstrate that a number line that begins at $\frac{2}{2}$ and ends at $\frac{7}{2}$, with five equal intervals, has intervals that are $\frac{1}{2}$, so the second partition represents $\frac{4}{2}$.

- Ask students to name a fraction on a number line given a point. Students use knowledge of the concepts of denominator and numerator, identify the whole, and add the number of unit fractions to determine the point in the diagram as $\frac{5}{8}$.



- Ask students to name the fraction on a number line given a point when the number line begins at a value that is not 0, begins at a value that is not 1, and/or requires that additional partitions are made. Students should determine the point in the diagram is $\frac{7}{8}$.



Key Academic Terms:

number line, partition, tick mark, fraction, numerator, denominator, divide, equal parts, whole, equal intervals, value

Additional Resources:

- Activity: Fractions as numbers on a number line
- Game: Fractions on the number line

15a

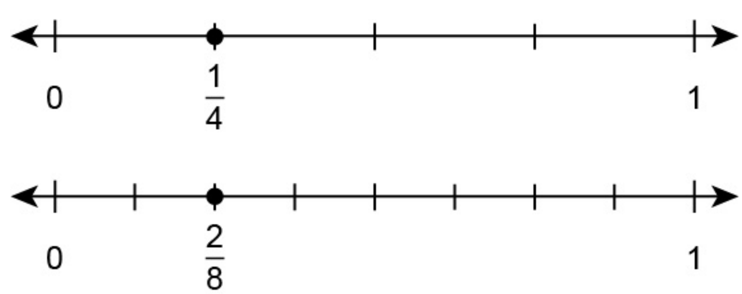
Operations with Numbers: Fractions
Develop understanding of fractions as numbers. *Denominators are limited to 2, 3, 4, 6, and 8.*
15. Explain equivalence and compare fractions by reasoning about their size using visual fraction models and number lines. a. Express whole numbers as fractions and recognize fractions that are equivalent to whole numbers.

Guiding Questions with Connections to Mathematical Practices:

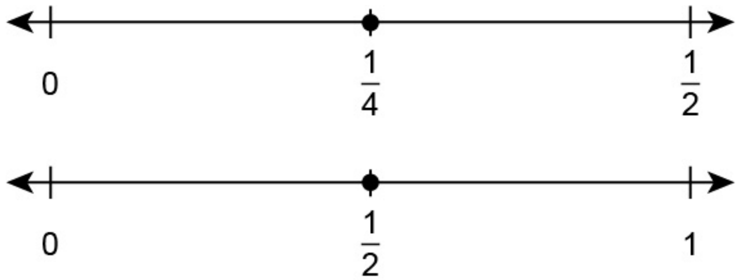
How are two or more fractions determined to be equivalent?

M.P.5. Use appropriate tools strategically. Explain with a variety of representations that two fractions can be the same size but have different names. For example, confirm that pattern blocks representing $\frac{1}{3}$ are exactly the same size as pattern blocks representing $\frac{2}{6}$; similarly, confirm that $\frac{1}{3}$ and $\frac{2}{6}$ are at the same location on a number line. Additionally, use a circle model with $\frac{2}{4}$ shaded to show that it is the same size as another equally sized circle with $\frac{4}{8}$ shaded or another equally sized circle with $\frac{3}{6}$ shaded.

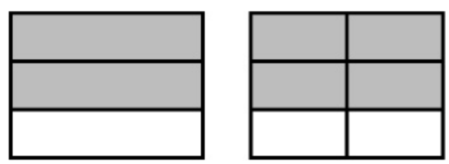
- Ask students to use two number lines with equal wholes to determine fraction equivalence. Students place a point on each number line to represent $\frac{1}{4}$ and $\frac{2}{8}$ to determine if the fractions are equivalent.



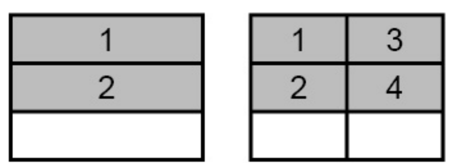
Students also determine equivalence given points on a number line, making sure to attend to the whole. Using the diagram shown, many students may incorrectly identify the fractions $\frac{1}{4}$ and $\frac{1}{2}$ as equivalent, by looking at the location of the number line without taking into account the whole.



- Ask students to determine equivalence using a variety of area models, including paper strips, circles, rectangles, and arrays. Count the number of shaded squares in the diagram to determine that $\frac{2}{3}$ is equivalent to $\frac{4}{6}$.



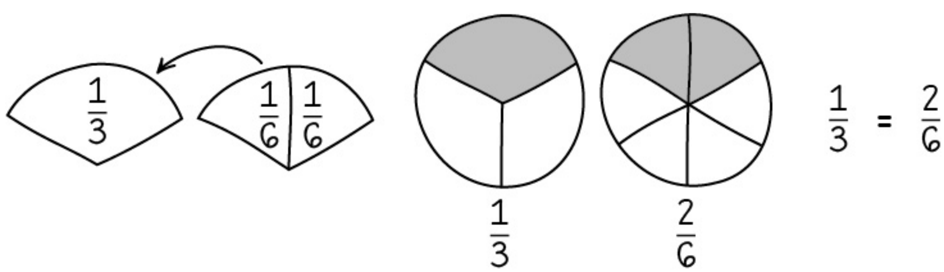
The students may also number the pieces in different ways to show equivalence when counting or skip-counting.



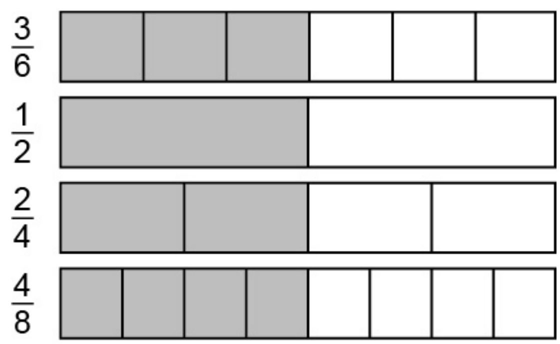
How can equivalent fractions be found for a given fraction?

M.P.6. Attend to precision. Explain and demonstrate how equivalent fractions can be found for a given fraction. For example, use a fraction strip folded into two parts with one part shaded to represent $\frac{1}{2}$. Fold the fraction strip in half again so there are four parts with two parts shaded. The fractions $\frac{1}{2}$ and $\frac{2}{4}$ both represent the same quantity of shading, so $\frac{1}{2}$ must equal $\frac{2}{4}$. Fold the fraction strip in half again to demonstrate $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$. Another example would be to use a letter fold to divide the shaded fraction strip so there are six parts with three parts shaded to show that $\frac{1}{2} = \frac{3}{6}$. Additionally, explain why the fractions are equivalent, providing statements such as "$\frac{1}{2}$ and $\frac{3}{6}$ have the same part of the whole shaded on the fraction strip."

- Ask students to create equivalent fractions using hands-on models, written representations, and equations, making connections between the fraction equivalence representations. In the diagram, students have used fraction circles to show that two $\frac{1}{6}$ pieces cover up $\frac{1}{3}$ exactly and made a drawing to represent the equivalence. Finally, write an equation to show that $\frac{1}{3} = \frac{2}{6}$.



- Ask students to generate equivalent fractions using hands-on models, written representations, and equations, making connections between the fraction equivalence representations. Add or remove partitions in order to find fraction equivalence. As shown below, use fraction paper strips to find fractions equivalent to $\frac{3}{6}$, including additional folds of the $\frac{3}{6}$ paper strip. Fold other paper strips and shade the strips to find fractions with denominators that are not multiples of 6. Then, make a drawing to represent the equivalence, and, finally, write an equation to show $\frac{3}{6} = \frac{1}{2} = \frac{2}{4} = \frac{4}{8}$. Statements like "One-and-a-half thirds are equivalent to one-half" are mathematically accurate and should be accepted, but not explicitly taught.

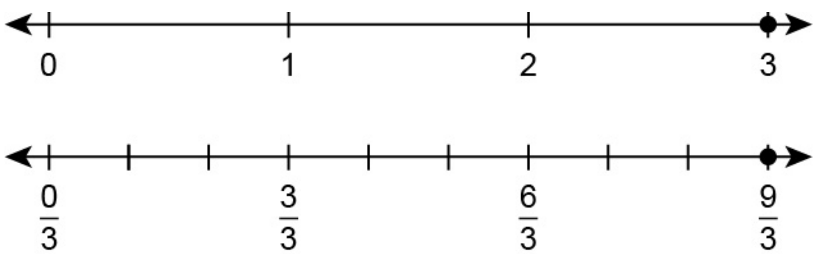


- Ask students to informally find and use patterns using multiplication in fraction equivalence, without formalizing this practice into a rule. Begin to generalize the connection between multiplication and fraction equivalence, making statements such as, "When I folded my fraction strip (showing $\frac{2}{4}$) in half again, the total number of pieces (denominator) doubled to 8, and the number of shaded parts (numerator) doubled to 4. Even though both the digits doubled, the area is the same between $\frac{2}{4}$ and $\frac{4}{8}$, so the fractions are equivalent."

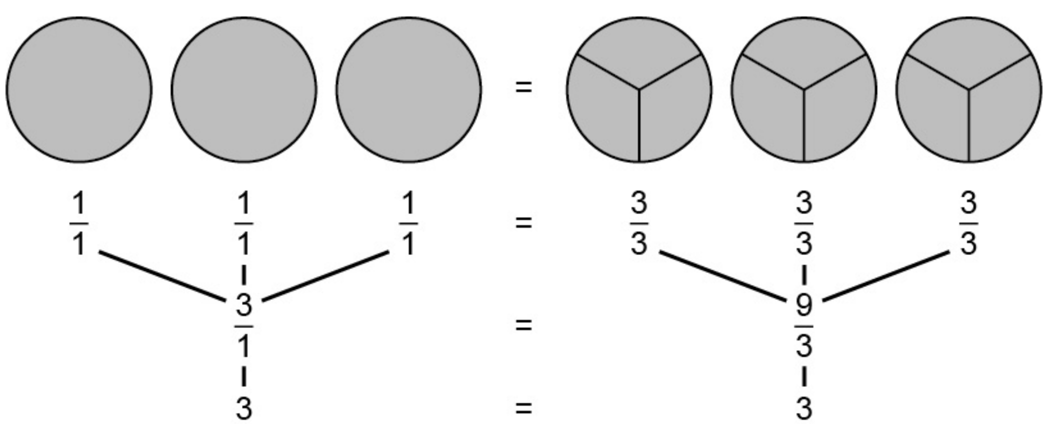
How are whole numbers written as fractions?

M.P.2. Reason abstractly and quantitatively. Write any whole number as an equivalent fraction and represent the fraction in a variety of ways. For example, $2 = \frac{4}{2}$, because 2 and $\frac{4}{2}$ are located at the same point on a number line. Also, know that $\frac{1}{1}$ is one part of a whole that is divided into 1 part and is equal to 1. Similarly, $\frac{2}{1}$ is "two wholes" and is equal to 2. Additionally, begin to see informally that whole numbers and their equivalent fractions are related by division.

- Ask students to use number lines and area models to show equivalence between fractions and whole numbers. The number line shows that 3 and $\frac{9}{3}$ are equivalent as they are at the same location on the number line.



Additionally, students use fraction circles to show that $\frac{3}{1}$, 3, and $\frac{9}{3}$ are all equivalent, when one circle is one whole.



- Ask students to use fraction language with a denominator of 1 being "whole." For example, the fractions $\frac{6}{1}$, $\frac{11}{1}$, and $\frac{7}{1}$ are read as "six wholes, eleven wholes, and seven wholes."

- Ask students to informally make conclusions and justify whole number and fraction equivalence. After repeated experience, students may informally begin to notice fractions as division. Students may state, “I noticed that $\frac{12}{3}$ is equal to 4, and $12 \div 3 = 4$. I wonder if that works for other fractions.”

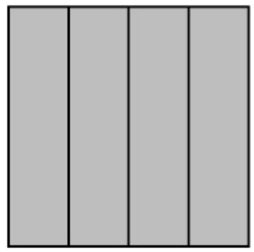
What fractions are equivalent to 1?

M.P.7. Look for and make use of structure. Know that any fraction that has the same nonzero numerator and denominator is equivalent to 1. For example, $\frac{3}{3}$ and $\frac{6}{6}$ are both equivalent to 1 and to each other, because they each represent 1 whole. Additionally, $\frac{1}{1}$, 1, and $\frac{8}{8}$ are all equivalent and equal to 1 whole.

- Ask students to decompose and compose fractions with the same nonzero numerator and denominator to show that those fractions are equivalent to 1. Students should conclude that $1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3}$.
- Ask students to form generalizations after repeated experiences with fractions with the same numerator and denominator to show that any fraction with the same nonzero numerator and denominator will be equal to 1. Students may state, “I noticed that $\frac{3}{3}$ is the same as 1, $\frac{2}{2}$ is the same as 1, and $\frac{6}{6}$ is the same as 1. All the fractions have the same number for the numerator and denominator and all equal 1. I wonder if that works for all fractions that have the same number for the numerator and denominator.”

M.P.5. Use appropriate tools strategically. Demonstrate the concept that any fraction that has the same nonzero numerator and denominator is equivalent to 1 by using regions and/or a number line. For example, show that pattern blocks representing $\frac{2}{2}$ are exactly the same size as pattern blocks representing $\frac{6}{6}$; similarly, verify that $\frac{3}{3}$ and $\frac{4}{4}$ are at the same location on a number line, which is also the location of 1 on a number line. Additionally, partition a rectangle into eight equal parts, shade in eight, and note that an entire whole is shaded so that it is equivalent to 1.

- Ask students to create fractions that are equivalent to 1 with different size denominators using fraction circles, fraction strips, area models, and written fraction form. In the diagram, students drew a rectangle to show that $\frac{4}{4}$ is four equal parts with four parts shaded, which is one whole.



- Ask students to use a variety of representations to explain how to create a fraction equal to 1 and that when given a fraction where the numerator and denominator are the same nonzero digit, the fraction is equivalent to 1. Students may state, "A whole is 1 circle. Three one-third pieces fill the circle completely. So, $\frac{3}{3}$ is the same as 1 whole."

Key Academic Terms:

equivalent, fraction, number line, point, compare, model, fraction strip, whole, numerator, denominator, nonzero

Additional Resources:

- Article: Real-world examples for teaching fractions on a number line
- Activity: Comparing fractions
- Activity: S'mores equivalent fractions game
- Video: Equivalent fractions
- Video: Equivalent fractions
- Lesson: Introducing equivalent fractions using equal sharing problems
- Lesson: Fractions equivalent to whole numbers
- Lesson: Initial fraction ideas

15b

Operations with Numbers: Fractions

Develop understanding of fractions as numbers.

Denominators are limited to 2, 3, 4, 6, and 8.

15. Explain equivalence and compare fractions by reasoning about their size using visual fraction models and number lines.

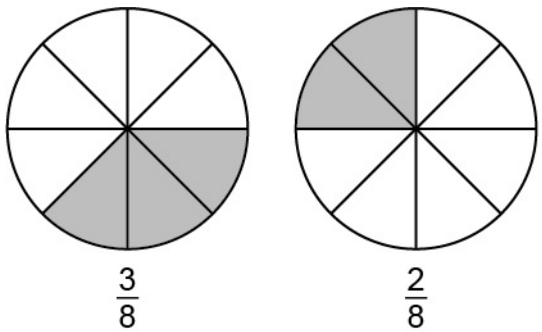
b. Compare two fractions with the same numerator or with the same denominator by reasoning about their size (recognizing that fractions must refer to the same whole for the comparison to be valid). Record comparisons using <, >, or = and justify conclusions.

Guiding Questions with Connections to Mathematical Practices:

How can fractions with the same denominator be compared?

M.P.3. Construct viable arguments and critique the reasoning of others. Use models to show that the numerator of a fraction indicates the number of parts, so if the denominators of two fractions are the same, the fraction with the greater numerator is the greater fraction. For example, in comparing $\frac{2}{6}$ to $\frac{1}{6}$, demonstrate with fraction models that $\frac{2}{6}$ is the greater fraction because it represents 2 of 6 parts, whereas $\frac{1}{6}$ represents 1 of those same 6 parts. Additionally, $\frac{1}{3}$ is a smaller fraction than $\frac{4}{3}$ because $\frac{1}{3}$ is to the left of $\frac{4}{3}$ on the number line.

- Ask students to use number lines, area models, fraction circles, and paper strips to compare fractions with different numerators and same denominators. In the diagram, students compare fraction circles to note that $\frac{3}{8}$ is more than $\frac{2}{8}$ because there is one more $\frac{1}{8}$-sized piece in $\frac{3}{8}$.

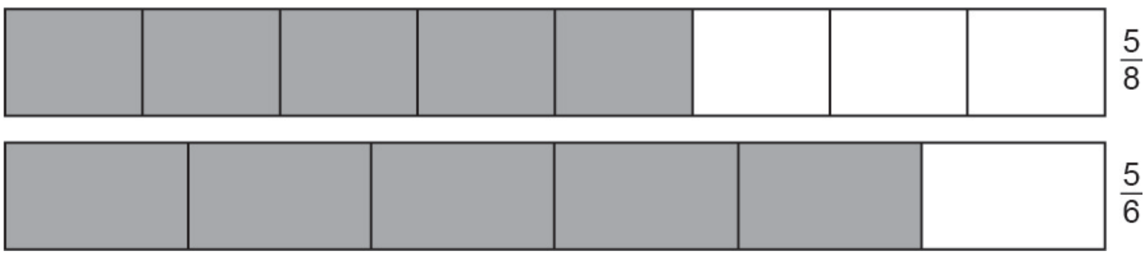


- Ask students to make informal justifications and generalizations about comparing fractions with the same denominator. After a variety of experiences, students should generalize to be able to know that when the denominator is the same, the fraction with the larger digit in the numerator is the greater number. They may also reason that since the parts are the same size, having more parts means that that fraction is larger.

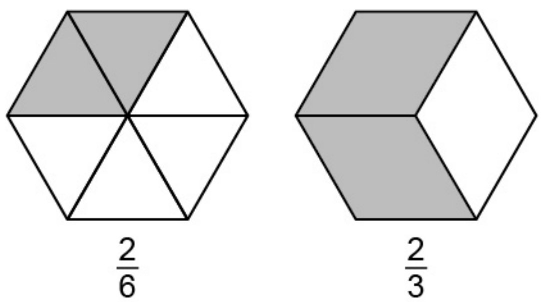
How can fractions with the same numerator be compared?

M.P.3. Construct viable arguments and critique the reasoning of others. Use models to show that the denominator of a fraction indicates the size of (equal) parts a whole is partitioned into, and that the greater the denominator, the smaller the parts. For example, in comparing $\frac{2}{3}$ to $\frac{2}{6}$, demonstrate with fraction models that $\frac{2}{3}$ is the greater fraction because both fractions have an equal number of parts, but thirds are larger than sixths. Additionally, demonstrate that the fraction $\frac{5}{8}$ is less than $\frac{5}{2}$ by using benchmark fractions: $\frac{5}{8}$ is a little more than $\frac{1}{2}$ and $\frac{5}{2}$ is more than 2.

- Ask students to use number lines, area models, fraction circles, and paper strips to compare fractions with same numerators and different denominators. Students should have conversations about the relation of the size of the piece in order to compare fractions with same numerators and different denominators. Students should use paper strips to show that $\frac{5}{8}$ is smaller than $\frac{5}{6}$ because sixths are larger than eighths, so 5 of a larger piece means more area covered than 5 smaller pieces.



Students may also use benchmark fractions, such as $\frac{1}{2}$ and 1. In the diagram, students use pattern blocks to show that $\frac{2}{6}$ is smaller than $\frac{2}{3}$ because $\frac{2}{6}$ is less than $\frac{1}{2}$ and $\frac{2}{3}$ is bigger than $\frac{1}{2}$.



- Ask students to make informal justifications and generalizations about comparing fractions with the same numerator. After a variety of experiences, students should be able to know that when the numerators are the same, the fraction with the smaller digit in the denominator is the greater number. They may also reason that since the number of parts are the same, the smaller denominator means bigger parts, therefore the smaller denominator means a larger fraction.

When is it appropriate/not appropriate to compare two fractions?

M.P.2. Reason abstractly and quantitatively. Determine whether fractions are describing the same whole before comparing them. For example, $\frac{3}{4}$ of a small sandwich is not the same as $\frac{3}{4}$ of a large sandwich because the whole is not the same. Additionally, $\frac{1}{2}$ of an array of 12 unit squares is not the same as $\frac{1}{2}$ of an array of 2 unit squares.

- Ask students to write contexts and draw pictures to show when fractions should or should not be compared. Students may state, "$\frac{1}{8}$ of a large piece of fruit is bigger than $\frac{1}{8}$ of a small piece of fruit, so the fractions are not the same $\frac{1}{8}$," and draw a picture of a large piece of fruit divided into eighths and a small piece of fruit divided into eighths to show the fractions are not the same.

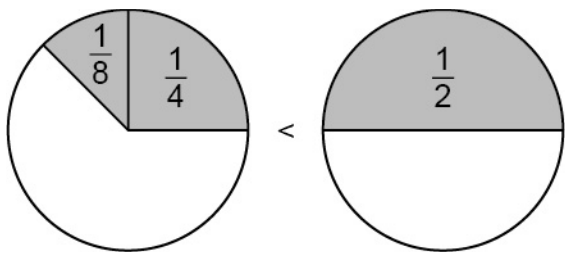
- Ask students to make justifications about the meaning of fractions and the importance of the whole when comparing, whether the fractions are equal or unequal. For example, when comparing fractions without context, such as $\frac{4}{2}$ and $\frac{4}{8}$, the whole is assumed to be the same, so $\frac{4}{2}$ is greater than $\frac{4}{8}$. When given a context, whether it is a story or drawing, it is essential to ensure that the wholes are the same before comparing the fractions to ensure the comparison is accurate.

How can symbols be used to record the comparison of two fractions?

M.P.6. Attend to precision. Record the comparison of two fractions using $>$, $=$, or $<$. For example, $\frac{2}{3} > \frac{2}{6}$. Additionally, $\frac{8}{4} = \frac{2}{1}$.

- Ask students to demonstrate that the comparison symbols $>$, $=$, or $<$ are used to create true statements. For example, $1 < 8$, therefore $\frac{1}{8} < \frac{8}{8}$.

- Ask students to write a missing fraction that is greater than/less than/equal to another fraction using symbols, e.g., $\square < \frac{1}{3}$ could have a correct solution of $\frac{1}{8}, \frac{1}{4}, \frac{1}{6}$ or an infinite number of other fractions. Choose fractions and symbols to write true comparisons, such as $\frac{6}{8} > \frac{5}{8}$. Decompose fractions in inequalities when comparing the fractions. For example, students may notice that a $\frac{1}{8}$ fraction circle piece and a $\frac{1}{4}$ fraction circle piece combined are less than a $\frac{1}{2}$ fraction circle piece, so $\frac{1}{8} + \frac{1}{4} < \frac{1}{2}$. Students would not need to know the exact sum of $\frac{1}{8}$ and $\frac{1}{4}$ in order to evaluate the inequality.



Key Academic Terms:

compare, fractions, numerator, denominator, part, whole, greater than (>), less than (<), equal (=), fraction models

Additional Resources:

- Video: [The best way to learn fractions!]
- Book: Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: fractions & decimals*. Portsmouth, NH: Heinemann.

16a

Data Analysis
Represent and interpret data.
16. For a given or collected set of data, create a scaled (one-to-many) picture graph and scaled bar graph to represent a data set with several categories. a. Determine a simple probability from a context that includes a picture.

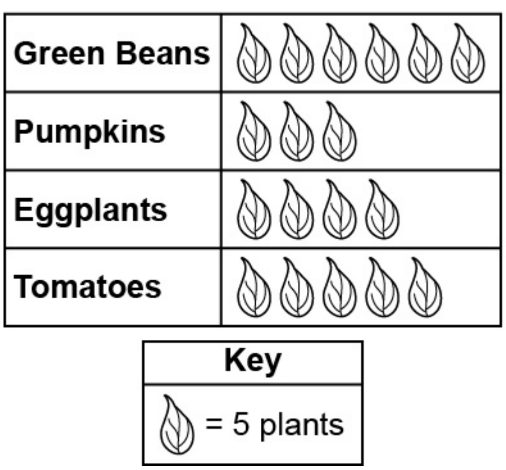
Guiding Questions with Connections to Mathematical Practices:

When drawing a bar graph or picture graph, what is “scale” and why is it important?

M.P.3. Construct viable arguments and critique the reasoning of others. Define scale and use it to create a graph. For example, when given data, create a picture graph where 1 picture of an insect represents 3 actual insects. Explain the importance of noting the scale when comparing or finding the actual values of the data representation. Additionally, make connections between the use of scale and multiplication or skip-counting.

- Ask students to determine how many items are represented by a picture graph when the value of each symbol is not one-to-one. In the example shown, students have correctly filled out a table from the given picture graph representing the number of vegetable plants in a garden.

Garden Plants

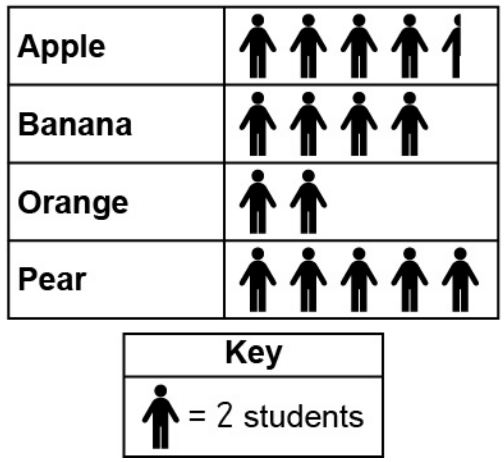


Green Beans	30
Pumpkins	15
Eggplants	20
Tomatoes	25

- Ask students to label the scale on a bar graph or picture graph that corresponds to data in a table. In the example shown, a data table of favorite fruits is given along with a picture graph that displays the data but without a scale, and a student has determined that the key for the graph should be 1 picture represents 2 students who chose a fruit.

Favorite Fruits

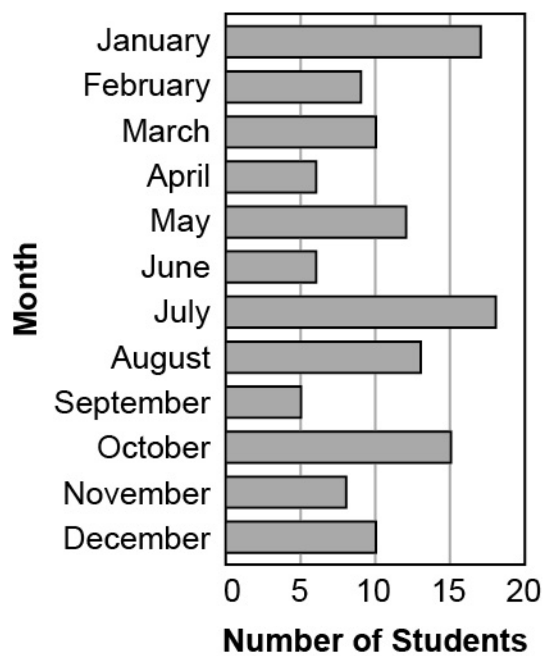
Apple	9
Banana	8
Orange	4
Pear	10



- Ask students to create a bar graph to represent a data set using a scale that is not 1. For example, students have represented the data in a table on birth month of students using a bar graph with a scale that counts by 5.

Birth Month of Students

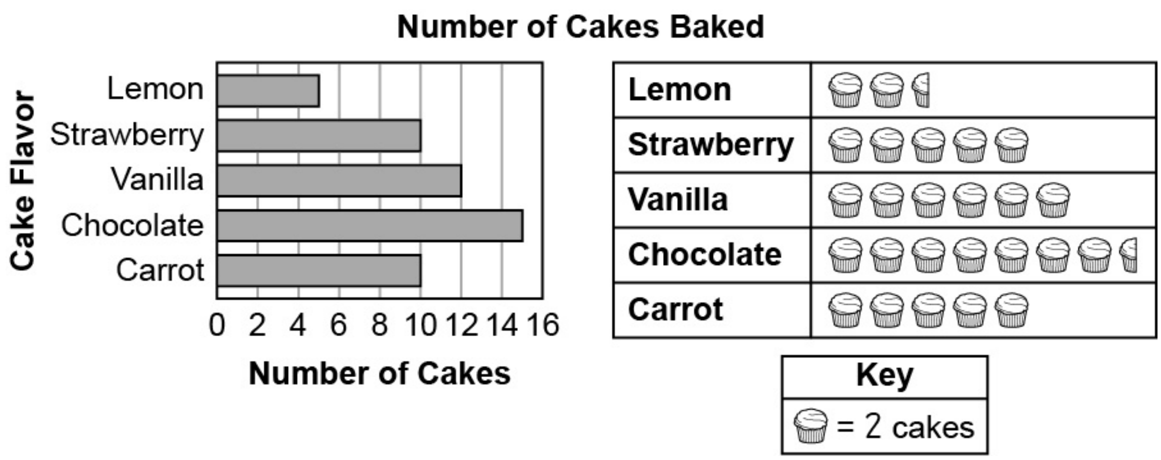
Month	Number of Students
January	𝍸 𝍸 𝍸 II
February	𝍸 IIII
March	𝍸 𝍸
April	𝍸 I
May	𝍸 𝍸 II
June	𝍸 I
July	𝍸 𝍸 𝍸 III
August	𝍸 𝍸 III
September	𝍸
October	𝍸 𝍸 𝍸
November	𝍸 III
December	𝍸 𝍸



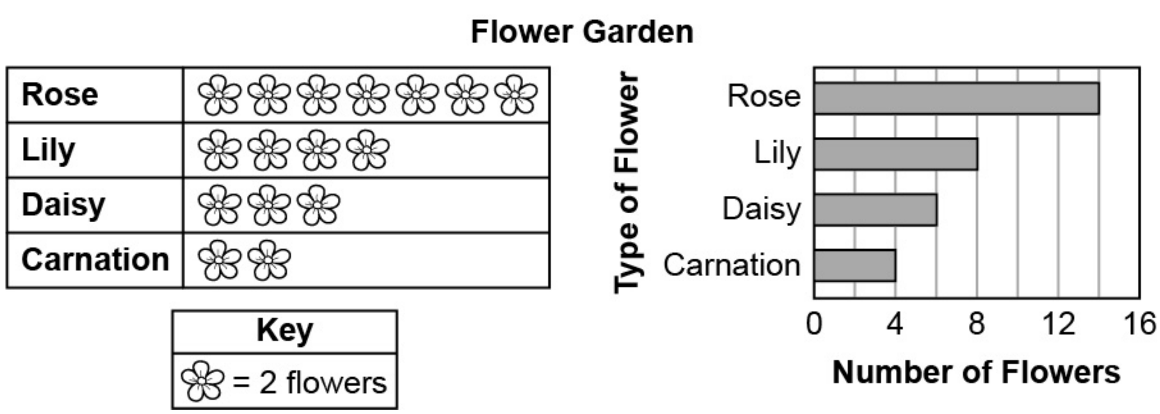
How are bar graphs and picture graphs connected?

M.P.7. Look for and make use of structure. Notice the similarities and differences between graphed data. For example, note that both bar graphs and picture graphs have a scale, and the scale for bar graphs is labeled on either the horizontal or vertical axis while the scale for picture graphs is given in a pictorial key. Additionally, both bar graphs and picture graphs are used to represent data and have a title and labels that explain the data.

- Ask students to use the scale on a bar graph to determine the key for a pictograph. In the following example, the bar graph shows a scale by steps of 2, so a student has determined that the key for the picture graph should also be 2.



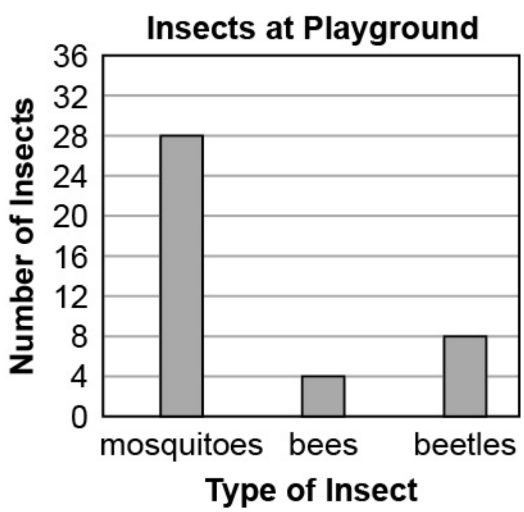
- Ask students to create a bar graph that displays the same data shown on a picture graph. For example, given a picture graph representing data on favorite type of flower, students have created a bar graph representing the same data.



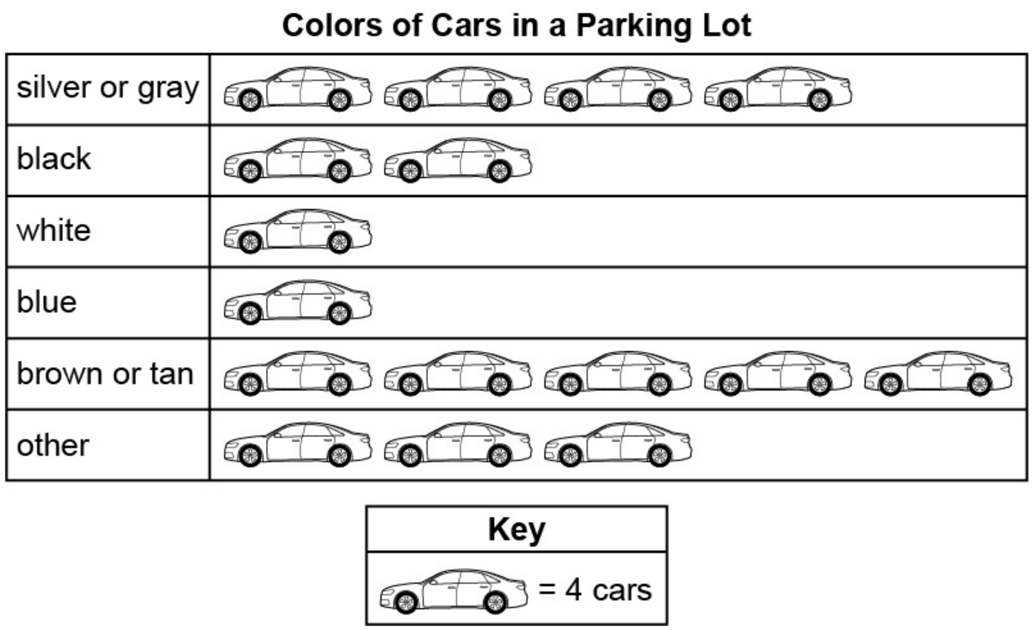
How can bar graphs or picture graphs be used to find probabilities?

M.P.4. Model with mathematics. Use a pictorial representation of data to find a simple probability. In bar graphs in which the bars represent the number of items in a category, a longer bar means there are more items in that category in the population than there are items represented with a shorter bar. For example, a bar graph has a longer bar showing how many students in a school play sports and a shorter bar showing how many students in the school sing in a choir. Therefore, if the name of a student is drawn out of a hat that includes the names of all students from the school, it is likely that the student plays a sport because more students play a sport than sing in a choir. Additionally, picture graphs can help students determine what is most probable or least probable by looking for the category that has the most or least pictures.

- Ask students to determine simple probabilities from a bar graph. For example, the bar graph shows the numbers of mosquitoes, bees, and beetles students saw at the playground during recess today. Ask students to determine which insect is most likely to be seen the most during recess tomorrow.



- Ask students to find simple probabilities from a picture graph. For example, the picture graph shows the numbers of cars in a parking lot and their colors. Ask students to find the most probable color of the next car entering the parking lot.



Students should note that even though the key shows that each picture is equal to 4 cars, the most probable color can still be found based on the number of pictures in each row.

Key Academic Terms:

bar graph, data, scale, picture graph, horizontal axis, vertical axis, pictorial key, probability

Additional Resources:

- Activity: Gummy bear graph
- Activity: Represent and interpret data

16b

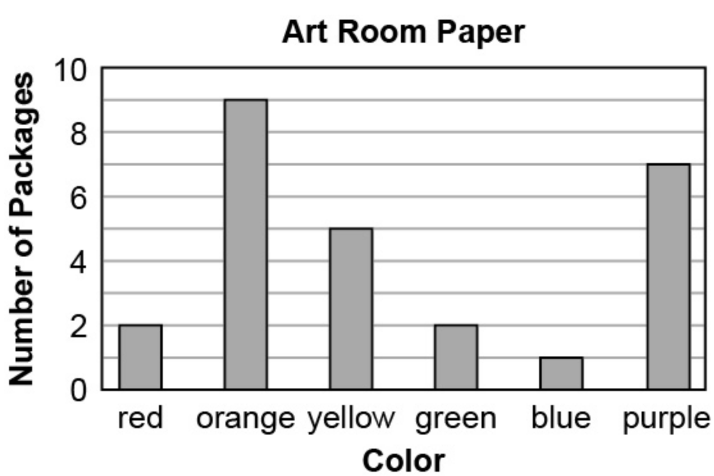
Data Analysis
Represent and interpret data.
16. For a given or collected set of data, create a scaled (one-to-many) picture graph and scaled bar graph to represent a data set with several categories. b. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled graphs.

Guiding Questions with Connections to Mathematical Practices:

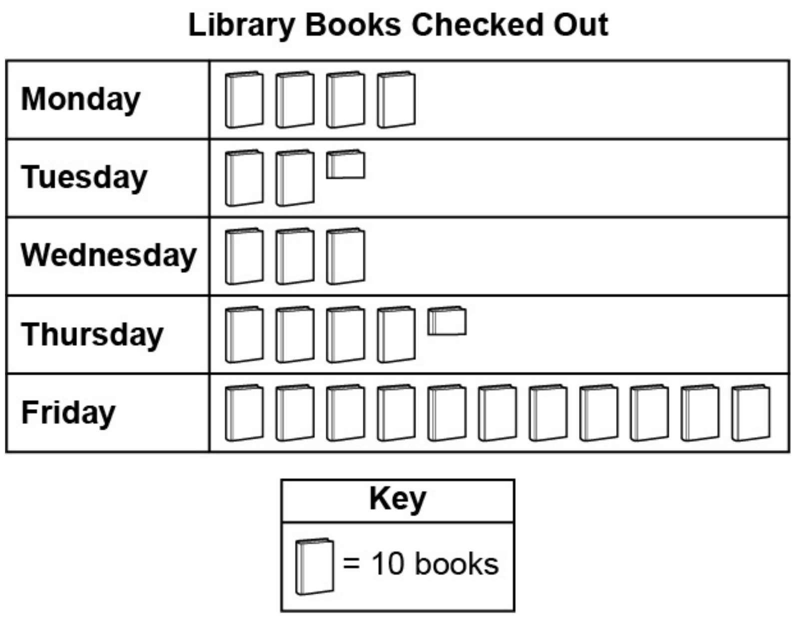
How can the data shown on a graph be used to solve problems involving quantities?

M.P.2. Reason abstractly and quantitatively. Interpret the relationships within data by solving problems to compare quantities. For example, if a bar graph shows that the most popular pet is a dog and the least popular pet is a bird, then the difference between the two bar lengths indicates how many more people prefer dogs than prefer birds. Additionally, demonstrate that the scale needs to be accounted for when examining the difference in bar lengths as well as the bar lengths themselves.

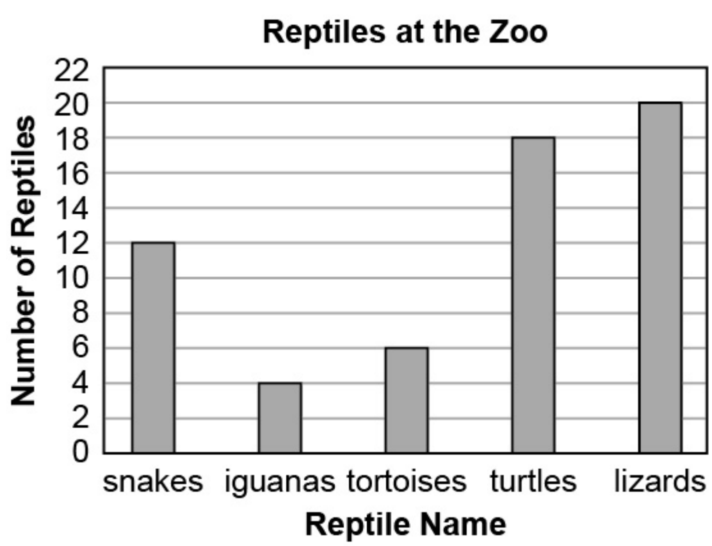
- Ask students to determine "how many more" or "how many less," given data in a bar graph. For example, students determine that there are 7 more packages of orange paper than red paper in the art room by subtracting 2 from 9.



- Ask students to determine "how many more" or "how many less," given data in a picture graph. For example, in a picture graph showing the number of books checked out from a library, it can be determined that there were 70 fewer books checked out on Monday than on Friday because 110 – 40 = 70. It can also be observed that because there are 7 fewer book icons on Monday than on Friday and the scale is 10, that there were 70 fewer books checked out on Monday than on Friday.



- Ask students to determine the sum of two or more categories in a bar graph or picture graph. In the example shown, it can be determined that a total of 60 reptiles are at the zoo because $12 + 4 + 6 + 18 + 20 = 60$. As an additional example, ask students to find how many more turtles and lizards there are than iguanas. Students first find that $18 + 20 = 38$ and then subtract 4 for the answer of 34.



Key Academic Terms:

bar graph, data, scale, picture graph, horizontal axis, vertical axis, pictorial key

Additional Resources:

- Activity: Gummy bear graph
- Activity: Represent and interpret data

17

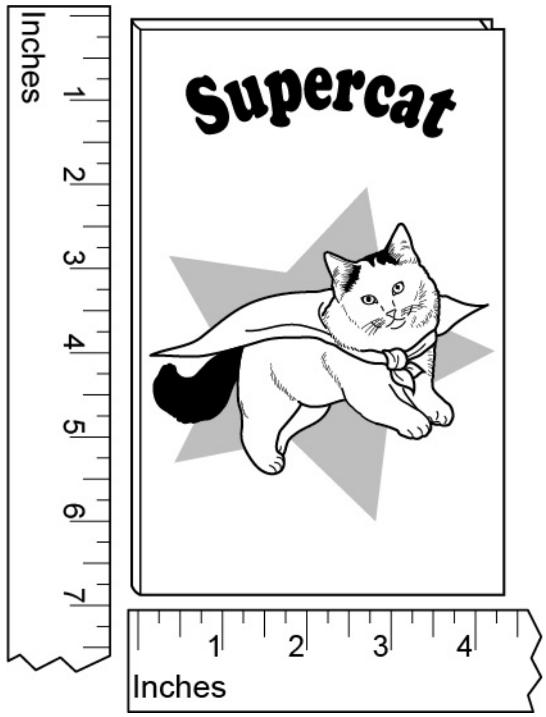
Data Analysis
Represent and interpret data.
17. Measure lengths using rulers marked with halves and fourths of an inch to generate data and create a line plot marked off in appropriate units to display the data.

Guiding Questions with Connections to Mathematical Practices:

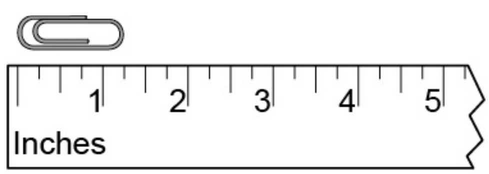
How is a ruler marked in halves and fourths of an inch used to measure length of an object?

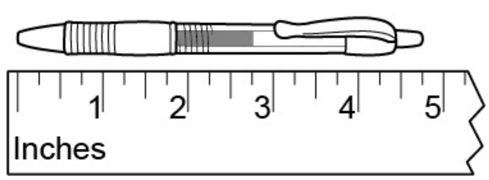
M.P.5. Use appropriate tools strategically. Confirm that the tick marks between two consecutive whole numbers on a ruler are equally spaced so that distances of halves and fourths can be measured. For example, the tick mark that is exactly halfway between the numbers 2 and 3 on a ruler indicates a length of $2\frac{1}{2}$ inches. Additionally, the tick mark that is exactly halfway between the 2-inch mark and the $2\frac{1}{2}$-inch mark indicates a length of $2\frac{1}{4}$ inches.

- Ask students to measure common objects to the nearest inch or half inch by determining which tick mark is closest to one end of the object when the other end of the object is placed at the first tick mark on the ruler. For example, measure a library book where the height is closest to 7 inches and the width is closest to $4\frac{1}{2}$ inches.



- Ask students to measure the lengths of common objects to the nearest quarter inch using a ruler marked with halves and fourths. For example, measure a paper clip at $1\frac{1}{4}$ inches and a pen at $4\frac{3}{4}$ inches.

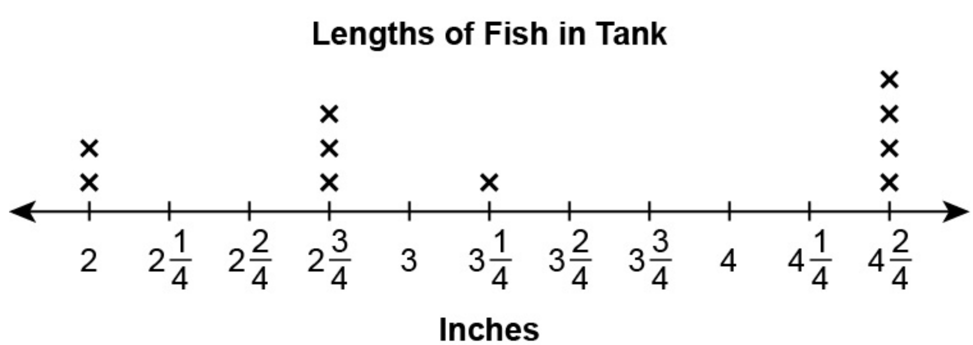




What is a line plot and how can it be used to show data involving whole numbers, halves, and quarters?

M.P.4. Model with mathematics. Demonstrate that a line plot is a graph that displays a distribution of data values, including whole numbers, halves, and quarters, such that each data value is marked above a horizontal line with an X or dot. For example, if a box of paper clips includes 3 clips that are each $\frac{2}{4}$ of an inch long, 4 clips that are each $\frac{3}{4}$ of an inch long, and 2 clips that are each 1 inch long, then a line plot can be constructed with 3 Xs stacked vertically above $\frac{2}{4}$, 4 Xs stacked vertically above $\frac{3}{4}$, and 2 Xs stacked vertically above 1. Additionally, if a line plot shows 6 Xs above the 3-inch mark and 5 Xs above the $3\frac{1}{4}$-inch mark, determine that there is 1 more clip measuring 3 inches than measuring $3\frac{1}{4}$ inches.

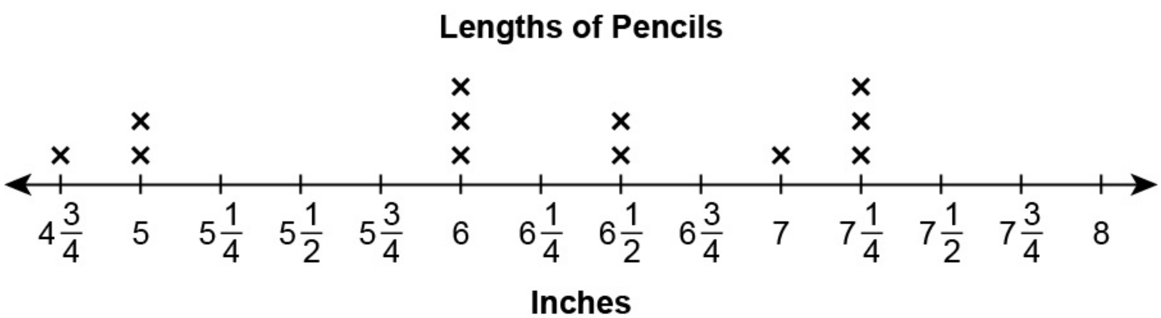
- Ask students to read information from a line plot. For example, given a line plot representing lengths of fish in a classroom tank, determine that there are 2 fish measuring 2 inches, 3 fish measuring $2\frac{3}{4}$ inches, 1 fish measuring $3\frac{1}{4}$ inches, and 4 fish measuring $4\frac{2}{4}$ inches.



- Ask students to create a line plot from a table of measurements. For example, measure a set of 12 pencils to the nearest quarter inch, record the data in a list, then transfer the data to a line plot.

Lengths of Pencils (inches)

$$7\frac{1}{4}, 6, 4\frac{3}{4}, 6\frac{1}{2}, 6, 7\frac{1}{4}, 6, 7, 6\frac{1}{2}, 5, 5, 7\frac{1}{4}$$



Key Academic Terms:

ruler, data, line plot, horizontal scale, distribution of data, halves, fourths, quarters

Additional Resources:

- Activity: Measure to the nearest half inch
- Book: Briggs, R. (1997). *Jim and the beanstalk*. New York, NY: Puffin Books. Activity

18a

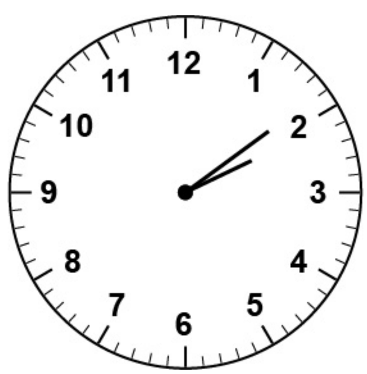
Measurement
Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
18. Tell and write time to the nearest minute; measure time intervals in minutes (within 90 minutes.) a. Solve real-world problems involving addition and subtraction of time intervals in minutes by representing the problem on a number line diagram.

Guiding Questions with Connections to Mathematical Practices:

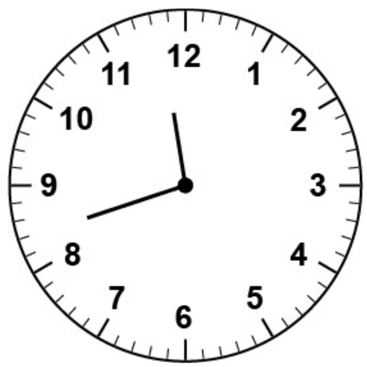
How is an analog clock used to tell time to the nearest minute?

M.P.6. Attend to precision. Explain that one of the wholes used to represent time is the circle of the clock face. The clock face is divided in two ways, by the hour and by the minute. The circular whole is divided into 12 parts where each part is 1 hour. The number of hours is represented by the short hand. Know that the hour is considered a whole that is also represented by the circle of the clock face. Each hour is divided into 60 parts and each part is 1 minute. The number of minutes is represented by the long hand. For example, if the short hand is between 6 and 7, and the long hand is 17 tick marks past 12, then the time is 6:17. Additionally, the clock face is divided into 12 sets of 5 minutes, and the groupings of 5 can be used to help count the number of minutes past the hour.

- Ask students to determine the time from a clock face to the nearest minute by counting the tick marks from 12. For example, show students the following clock face. Since the short hand (hour hand) is between the 2 and 3, and the long hand (minute hand) is 9 tick marks past the 12, the time is 2:09.

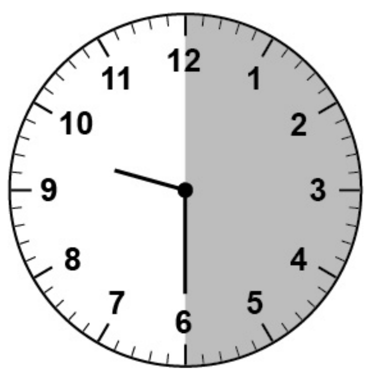


- Ask students to determine time from a clock face to the nearest minute using groups of 5 tick marks. For example, show students the following clock face. Since the short hand is between the 11 and 12, the hour is 11. The long hand is 2 tick marks past the 8. Skip-counting by 5, eight times, gives 40 minutes, plus the 2 additional tick marks gives a total of 42 minutes. Therefore, the time is 11:42.

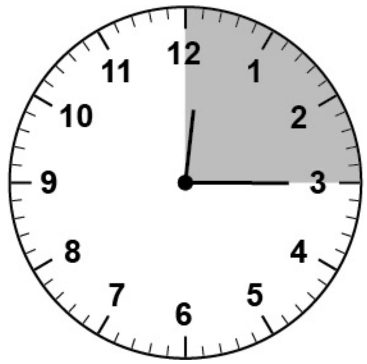


M.P.6. Attend to precision. Explore relationships within the clock to tell time, paying particular attention to using fractions on the clock face. For example, illustrate that when the minute hand is at the 3, it is 15 minutes, or a quarter of the whole, past the hour. Additionally, when the minute hand is at the 9, it is three-quarters of the whole, or 45 minutes past the hour.

- Ask students to interpret the time on a clock face using halves of the hour. For example, if the short hand is between the 9 and 10, and the long hand is at the six, the time is 9:30 because 30 minutes is half of the whole hour.



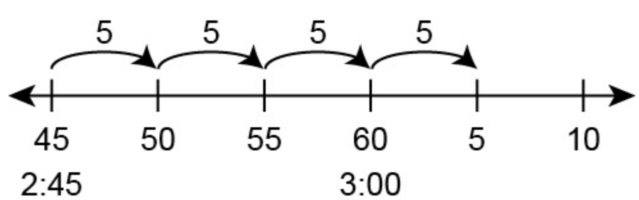
- Ask students to interpret the time on a clock face using quarters of an hour. For example, if the long hand is between the 12 and 1, and the short hand is on the 3, the time is 12:15 because it is one quarter hour, or 15 minutes, past 12.



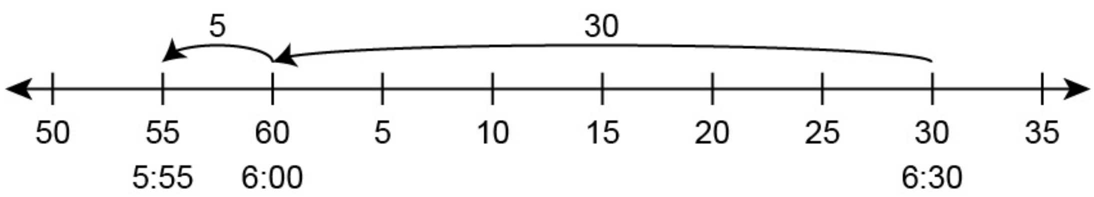
How can a number line be used to solve word problems that involve time intervals?

M.P.5. Use appropriate tools strategically. Skip-count backward or forward on a number line to determine the time length of an event or the time an event begins or ends. For example, if Jose had a 25-minute trumpet lesson that ended at 4:15, then the time the lesson began can be determined by creating an open number line that starts at 4:15, doing a left jump of 15 and marking that as 4:00, then doing another left jump of 10 and marking that as 3:50, paying attention to the fact that 4:00 is equivalent to 3:00 plus 60 minutes. Additionally, if Nikki plans to read for 40 minutes and begins at 6:50, her ending time can be determined by marking 6:50 on a number line, jumping right 10 minutes to mark 7:00, then an additional 30 minutes to the right to reach 7:30, paying attention to the fact that 7:00 is 60 minutes past 6:00.

- Ask students to count forward on a number line to determine the ending time of an event. For example, Titus is sitting in a 20-minute presentation that started at 2:45, and the ending time of the presentation can be determined by skip-counting forward on a number line by increments of 5 minutes from 2:45 until 3:05.



- Ask students to count backward on a number line to determine the length of time between events. For example, if a show is set to begin at 6:30 and the current time is 5:55, the time until the start of the show is 35 minutes because it is 30 minutes from 6:30 back to 6:00 and then an additional 5 minutes back to 5:55.



Key Academic Terms:

analog clock, interval, number line, quarter of the whole, quarter past the hour, tick mark

Additional Resources:

- Video: Telling time to the nearest minute
- Article: 5 hands-on ways to teach telling time

19a

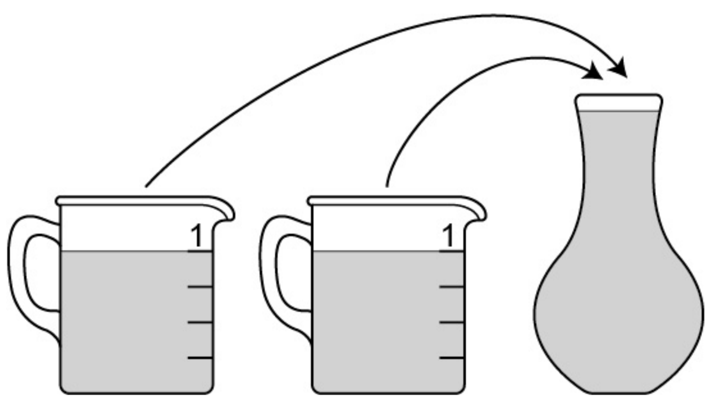
Measurement
Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
19. Estimate and measure liquid volumes and masses of objects using liters (l), grams (g), and kilograms (kg). a. Use the four operations to solve one-step word problems involving masses or volumes given in the same metric units.

Guiding Questions with Connections to Mathematical Practices:

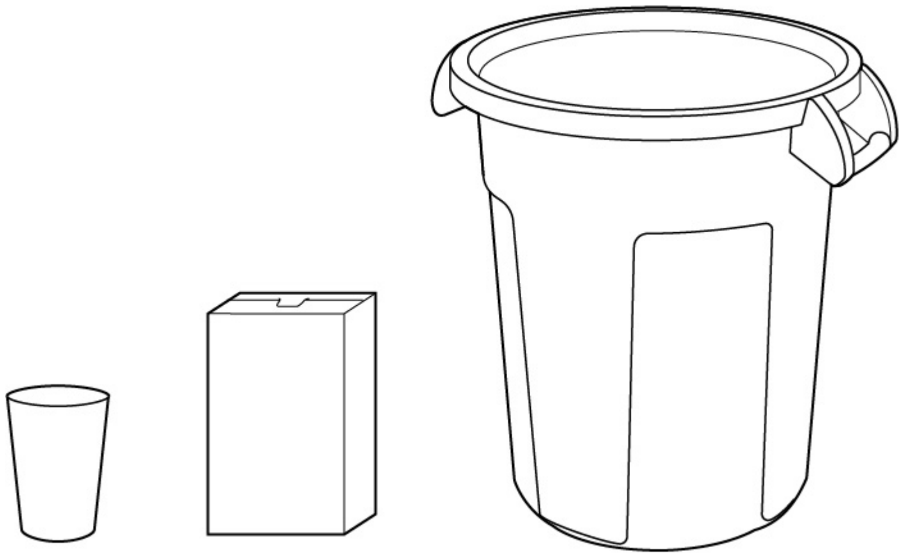
What is capacity and how is it measured?

M.P.2. Reason abstractly and quantitatively. Know that capacity indicates the measure of volume (dry or liquid) in a container. For example, the capacity of a jar is 1 liter if it can hold 1 liter of a pourable substance. Additionally, know that capacity needs to be measured using a pourable substance because the container needs to be filled without gaps.

- Ask students to measure the capacity of a container. For example, given an empty vase and two beakers of 1 liter of water each, students determine that the capacity of the vase is 2 liters because when the beakers are emptied into the vase, the vase is filled to capacity.



- Ask students to estimate the relative capacity of a variety of containers. For example, given a drinking glass, an empty cereal box, and an empty trash can, students order the items from least to greatest capacity.

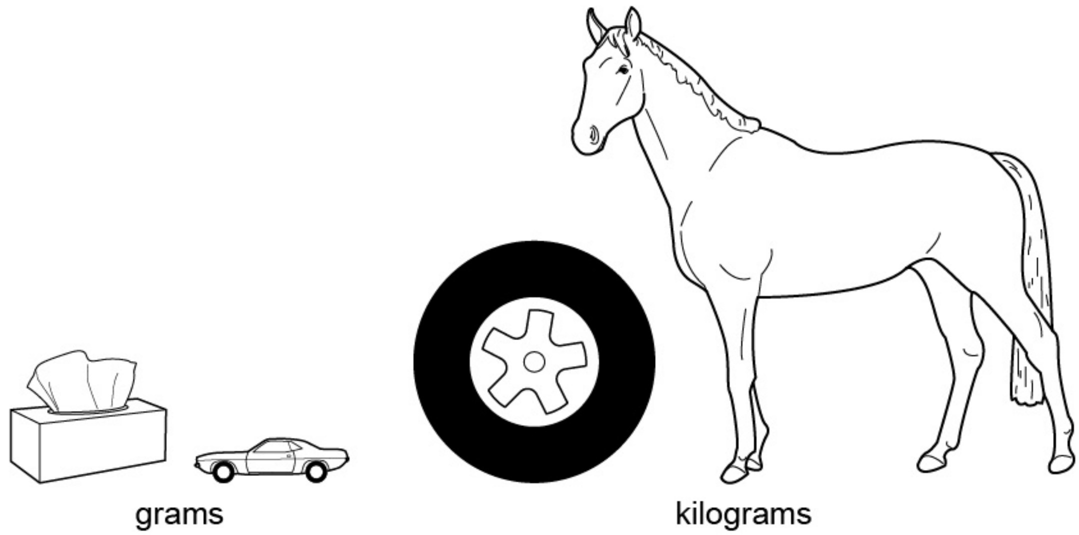


What is mass and how is it measured?

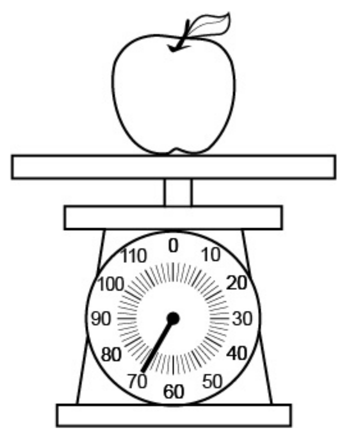
M.P.2. Reason abstractly and quantitatively. Know that mass indicates the amount of matter in an object and can be represented with units of different sizes, such as grams or kilograms. For example, the mass of smaller objects, such as pencils or paper clips, is generally recorded in grams, while the mass of larger objects, such as desks or doors, is recorded in kilograms. Additionally, the mass of an object is closely related to its weight and can be found using a tool such as a scale.

- Ask students to determine which of two objects has a greater mass and which has a greater volume. For example, ask students to compare a balloon and a brick. The balloon has a greater volume because it takes up more space. However, the brick has a greater mass because it contains more matter. As an additional example, ask students to compare an inflated balloon and a deflated balloon. The inflated balloon occupies a greater volume because it takes up more space. However, both the inflated and the deflated balloon have equal mass because they both contain the same amount of material. Similarly, a sponge that has been compressed has a smaller volume than it did before being compressed, but the mass is still the same.

- Ask students to determine which measure of mass would be best in certain cases. Given several objects or pictures of several objects, students determine which objects would best be measured in grams and which would best be measured in kilograms. For example, a toy car and a box of tissues would both be measured in grams, whereas a horse and a tire would both be measured in kilograms.



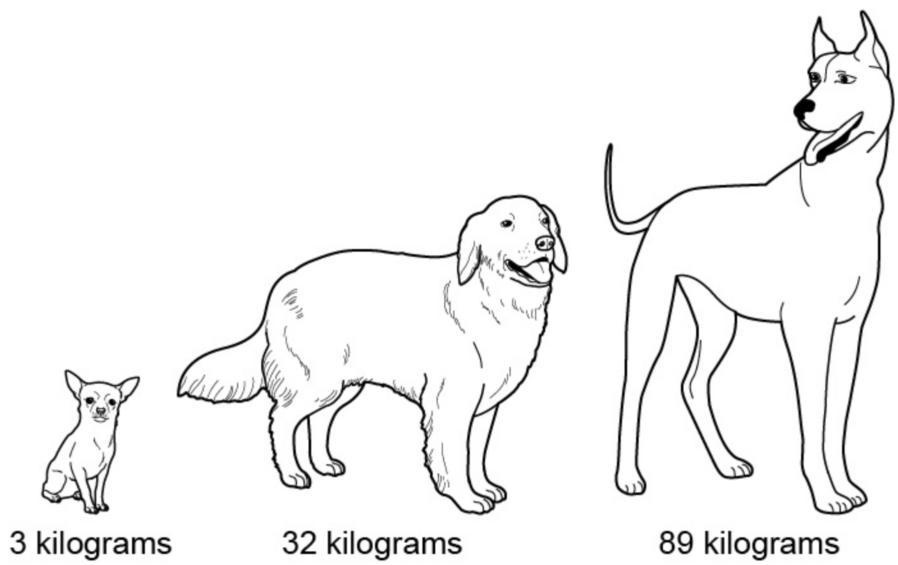
- Ask students to read a scale to determine the mass of an object. For example, the scale shows an apple with a mass of 70 grams.



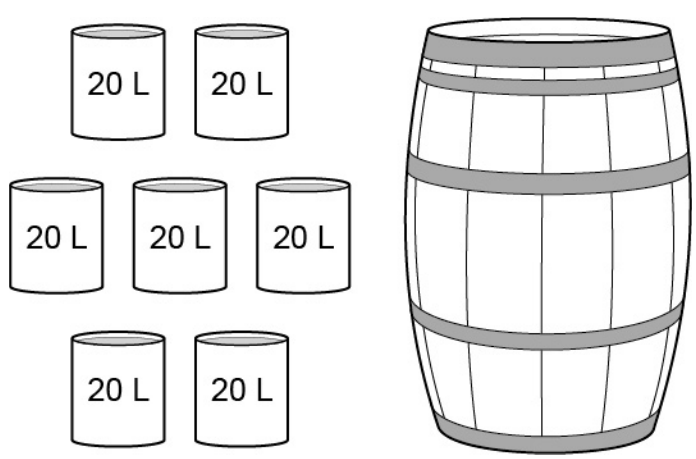
How can problems involving masses or volumes with the same unit be solved?

M.P.2. Reason abstractly and quantitatively. Extend previous knowledge of the four operations to add, subtract, multiply, and divide measures of volume and mass. For example, if the mass of one shoe is 335 grams, then the mass of the pair of shoes is 670 grams because $335 + 335 = 670$. Additionally, if the mass of a book bag is 960 grams, and a book weighing 85 grams is removed, the mass of the bag is 875 grams because $960 - 85 = 875$.

- Ask students to determine the total mass of multiple objects in a group. For example, if the masses of three dogs are 3 kilograms, 32 kilograms, and 89 kilograms, the total mass of the group of dogs is 124 kilograms because $3 + 32 + 89 = 124$.



- Ask students to determine volume resulting from multiplication. For example, if 7 buckets of water, each holding 20 liters, can be poured to fill a large, empty barrel, the volume of the barrel is 140 liters because $7 \times 20 = 140$.



Key Academic Terms:

volume, capacity, mass, matter, liter, gram, kilogram

Additional Resources:

- Activity: More or less than a liter?
- Activity: How heavy?

20

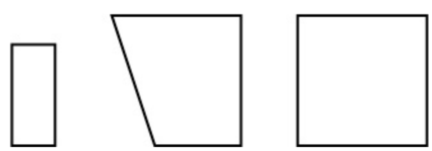
Measurement
Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
20. Find the area of a rectangle with whole number side lengths by tiling without gaps or overlays and counting unit squares.

Guiding Questions with Connections to Mathematical Practices:

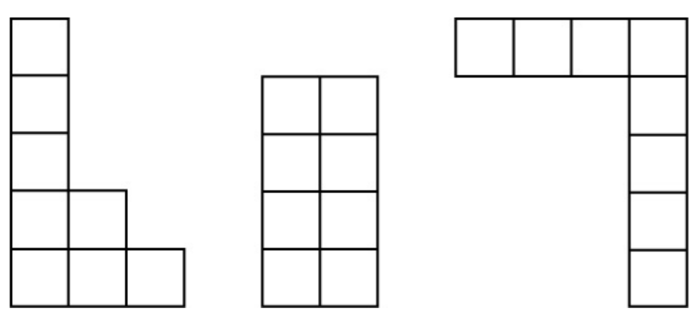
What is area?

M.P.7. Look for and make use of structure. Demonstrate that area is a measure of the size of a surface. For example, the area of a piece of paper can be measured by counting or calculating the number of identical squares required to cover one side of the paper. Additionally, area can be conceptualized as the amount of paint needed to cover a space; the larger the space, the more paint is required. More paint required means a greater area measurement.

- Ask students to determine relative area among figures. For example, given a set of three common shapes, order the areas of the shapes from least to greatest.



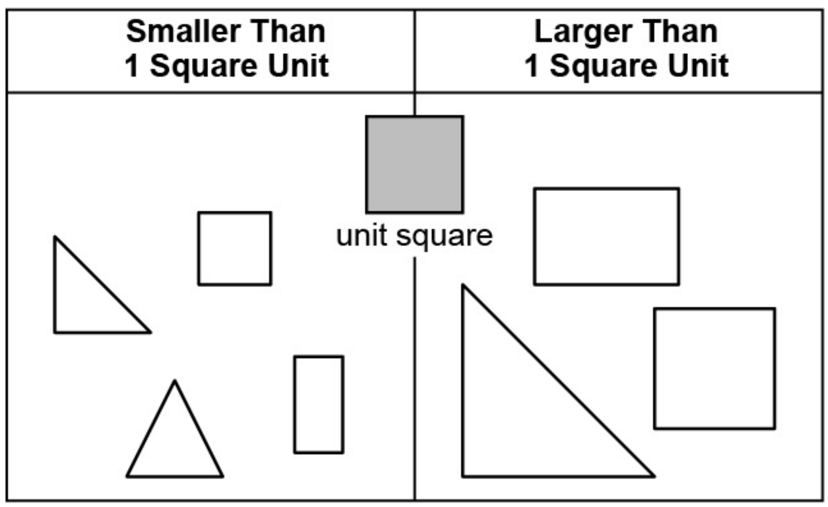
- Ask students to create figures of equal area using manipulatives. For example, construct three distinct figures with equal areas by arranging square tiles into solid shapes.



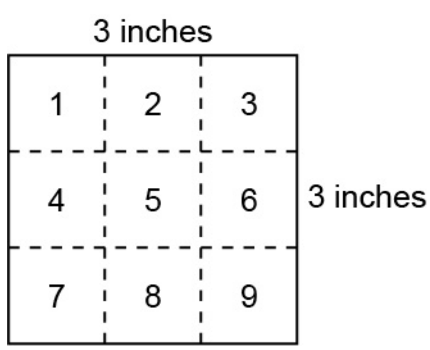
What is a unit square and how can it be used to measure area?

M.P.2. Reason abstractly and quantitatively. Know that a unit square is a square with a side length of 1 unit, and that such a square represents a unit of measurement. For example, if the surface area of a plane figure is exactly the same size as the surface area of a square with a side length of 1 unit, then the shape has an area of 1 square unit. Additionally, a square with a side length of 1 inch has an area of 1 square inch in the same way that a square with a side length of 1 centimeter has an area of 1 square centimeter. These squares are different sizes but are each unit squares.

- Ask students to determine whether figures have an area that is larger or smaller than a unit square. For example, cut a unit square out of paper to determine which shapes have an area that is smaller than the area of a unit square and which have an area that is larger than the area of a unit square.



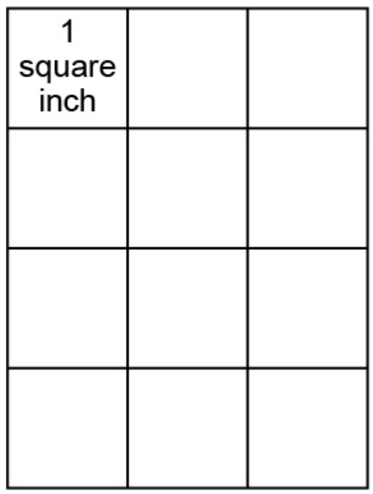
- Ask students to determine the area of larger squares using unit squares that measure 1 square inch. For example, given a square with a side length of 3 inches, determine that the area of this square is 9 square inches because it can be decomposed into 9 unit squares.



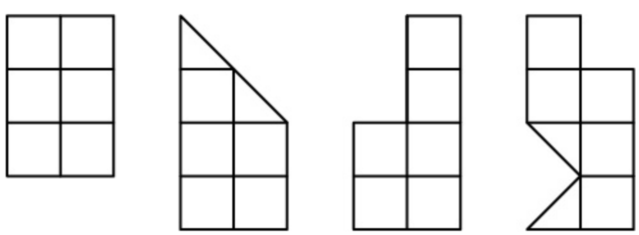
How can multiple unit squares be used to measure area?

M.P.5. Use appropriate tools strategically. Demonstrate that area is the number of unit squares needed to cover a surface, and demonstrate that multiple unit squares can be combined to measure the area of plane figures so long as the unit squares completely cover the figure without overlapping each other or extending beyond the edge of the figure. For example, if the surface of a flat rectangular ruler can be covered with 12 squares that each have a side length of one unit, then the ruler has an area of 12 square units. Additionally, unit squares can be "cut" in order to cover different types of shapes.

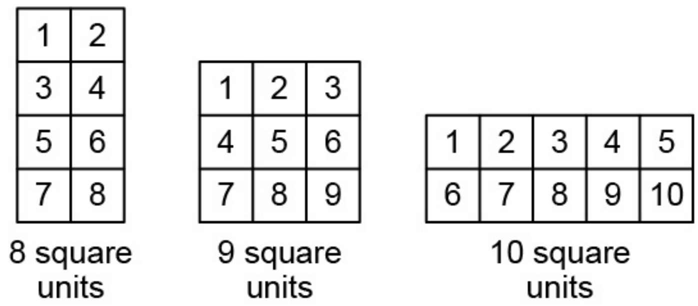
- Ask students to compose figures using unit squares with known units to define the area of the figure. For example, use a ruler to construct 12 squares, each with an area of 1 square inch, and by connecting the squares, with no gaps or overlaps, create a rectangle with an area of 12 square inches.



- Ask students to create figures with a given area by rearranging unit squares using graph paper or other manipulatives. For example, show a figure with an area of 6 square units, then create different figures that also each have an area of 6 square units by rearranging the 6 unit squares. Additionally, unit squares can be cut in half and the halves can be rearranged separately.



- Ask students to determine area by counting the unit squares that compose the figure. For example, label rectangles with an area measurement equal to the number of unit squares that compose the rectangle.



Key Academic Terms:

area, unit square, surface, attribute, plane figure, square unit, overlap, edge

Additional Resources:

- Worksheets: Area of shapes by counting unit squares
- Activity: The square counting shortcut
- Game: Counting squares to find area

21

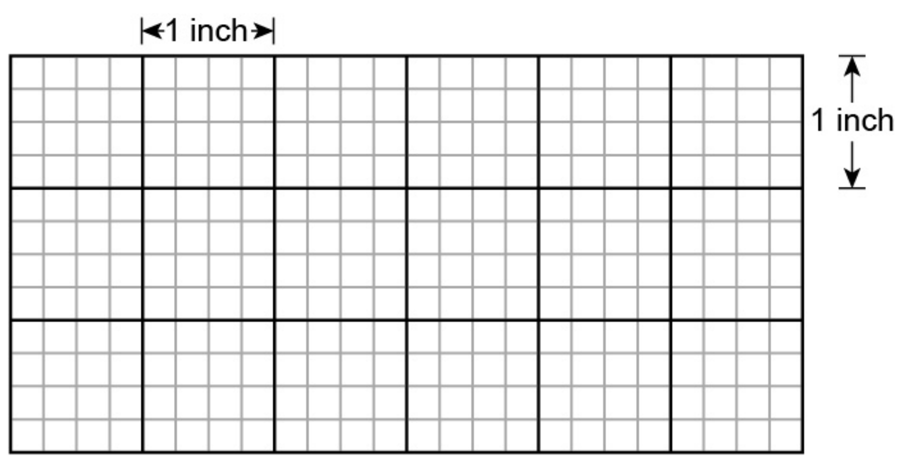
Measurement
Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
21. Count unit squares (square cm, square m, square in, square ft, and improvised or non-standard units) to determine area.

Guiding Questions with Connections to Mathematical Practices:

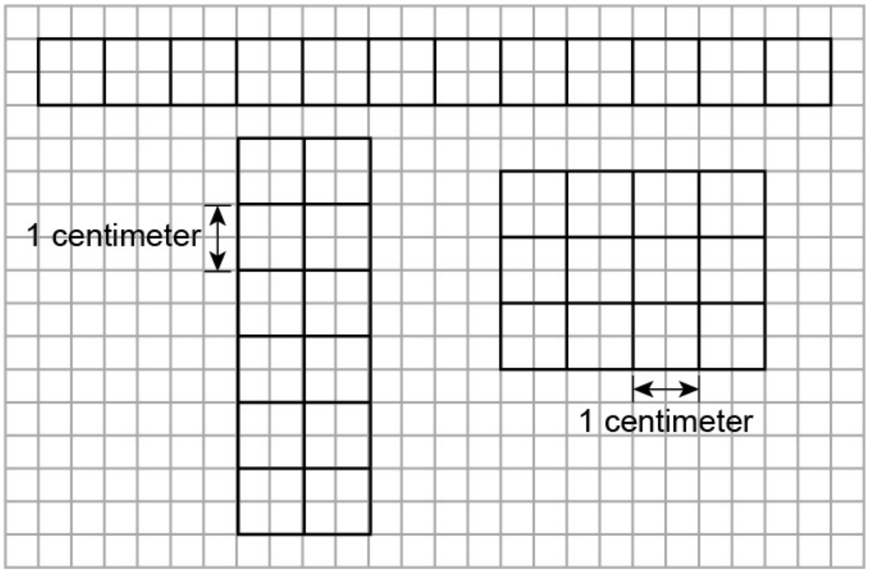
How can graph paper with square grids be used to measure the area of plane figures?

M.P.5. Use appropriate tools strategically. Determine the area of plane figures drawn on graph paper by counting the number of squares within the figure and draw figures of a given area. For example, if a rectangle made of 2 rows of 4 squares is drawn on graph paper, then the area of the rectangle is 8 square units because there are 8 total complete squares within the rectangle. Additionally, the unit squares being counted are not necessarily the squares on the graph paper.

- Ask students to count the number of unit squares within a given figure on graph paper. For example, give students the following figure shown on quarter-inch graph paper. Determine that in order to measure the area in square inches, each unit square should be 4 units by 4 units. The area of the figure shown is 18 square inches because it is made of 3 rows of 6 1-inch squares each having an area of 1 square inch.



- Ask students to draw figures of a given area on graph paper. For example, draw multiple rectangles with an area of 12 square centimeters on graph paper scaled with half centimeters. In order to measure area in square centimeters, each unit square should be 2 units by 2 units.



Key Academic Terms:

area, unit square, plane figure, square centimeters, square meters, square inches, square feet

Additional Resources:

- Worksheets: [Area of shapes by counting unit squares]
- Game: [Counting squares to find area]

22

Measurement
Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
22. Relate area to the operations of multiplication using real-world problems, concrete materials, mathematical reasoning, and the distributive property.

Guiding Questions with Connections to Mathematical Practices:

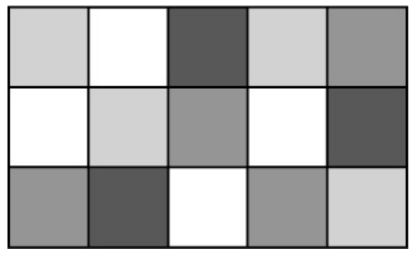
What is tiling and how is it used to determine the area of a rectangle with whole-number side lengths?

M.P.4. Model with mathematics. Illustrate tiling as the covering of an entire plane figure with nonoverlapping regular shapes, and that if the shapes are unit squares, then the total number of squares covering the plane figure will represent the area of the figure. For example, if a rectangle with side lengths of 3 inches and 6 inches is “tiled” into unit squares by dividing it into 3 equally sized columns and 6 equally sized rows, then the area of the rectangle is 18 square inches because there is a total of 18 one-inch unit squares. Additionally, a rectangle composed of 12 one-inch tiles may have dimensions of 1 inch by 12 inches, 2 inches by 6 inches, or 3 inches by 4 inches.

- Ask students to tile a rectangle with unit squares and determine the area of the rectangle by counting the square tiles. For example, given that a rectangle has a length of 7 centimeters and a width of 2 centimeters, draw 7 columns and 2 rows to determine that the area of the rectangle is 14 square centimeters by counting the number of square tiles.

1	2	3	4	5	6	7
8	9	10	11	12	13	14

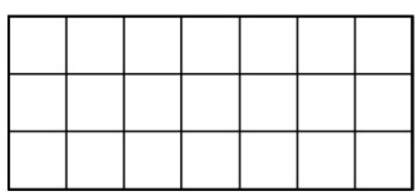
- Ask students to determine the length, width, and area of a rectangle using unit tiles. In the example shown, students have used square-inch manipulatives to tile a rectangle, determining that the width of the rectangle is 3 inches, the length is 5 inches, and the area is 15 square inches because the rectangle is composed of 3 rows of 5 tiles each.



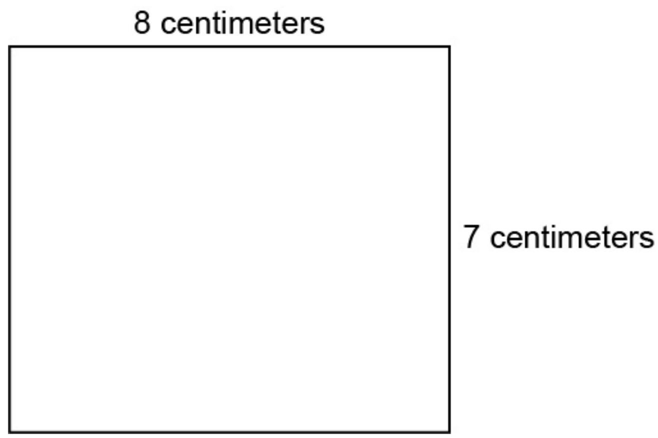
How is tiling a rectangle that has whole-number side lengths related to arrays and multiplication?

M.P.2. Reason abstractly and quantitatively. Demonstrate that tiling a rectangle that has whole-number side lengths with unit squares produces rows and columns, and that multiplying the number of rows by the number of columns is equivalent to the total number of squares just like arrays. For example, if a rectangle with side lengths of 5 inches and 2 inches is divided into 5 columns and 2 rows, then there is an array of 2 rows of 5 squares, which is the same as multiplying 2×5 or counting the squares one at a time. Additionally, the rectangle with an area of 10 square inches can also be represented with an array of 1 row and 10 columns or 1 column and 10 rows.

- Ask students to tile a rectangle with a specified length and width and determine the area of the rectangle using multiplication. For example, given that a rectangle has a width of 3 centimeters and a length of 7 centimeters, draw 3 rows and 7 columns and determine that the area of the rectangle is 21 square centimeters because the product of 3 and 7 is 21.



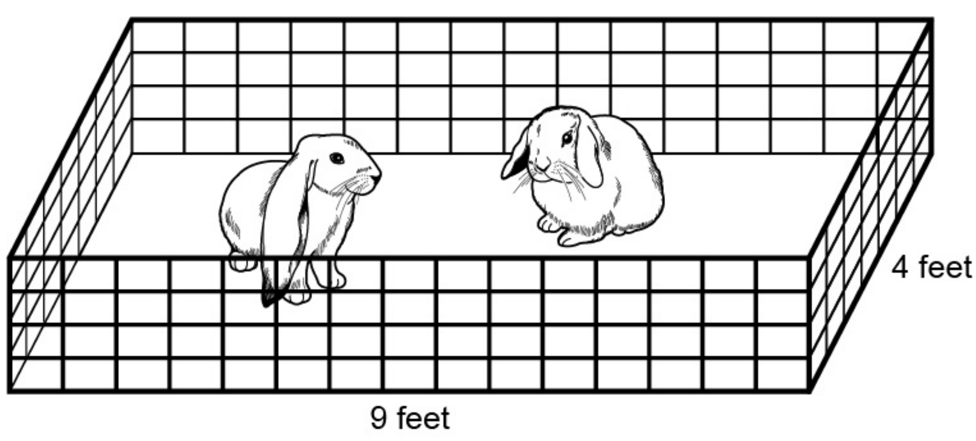
- Ask students to use multiplication to determine the area of a rectangle when given a specified length and width. For example, given a rectangle with length 8 centimeters and width 7 centimeters, the area can be calculated as $8 \times 7 = 56$ square centimeters.



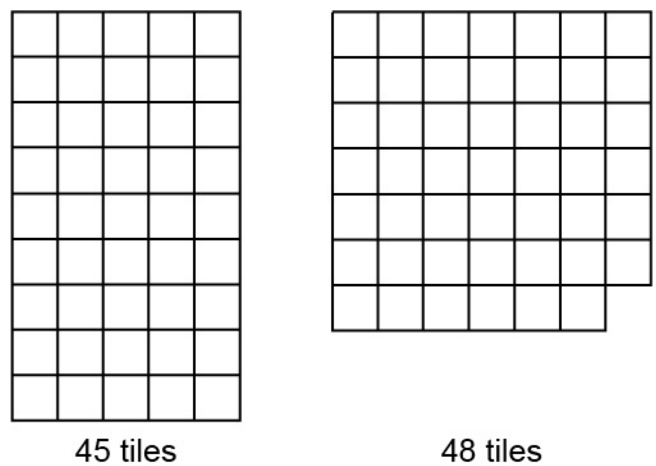
How can multiplication be used to determine the area of rectangles?

M.P.2. Reason abstractly and quantitatively. Extend previous experience with multiplication to determine the area of rectangular figures by finding the product of the two side lengths. For example, if the side lengths of a rectangular doormat are 2 feet and 3 feet, then finding the product of 2 and 3 can be used to determine that the doormat has an area of 6 square feet. Additionally, if a 16 square foot rectangular window is to be installed in a wall that measures 10 feet tall, two possibilities, when using whole numbers, are a 4-foot by 4-foot window and an 8-foot by 2-foot window.

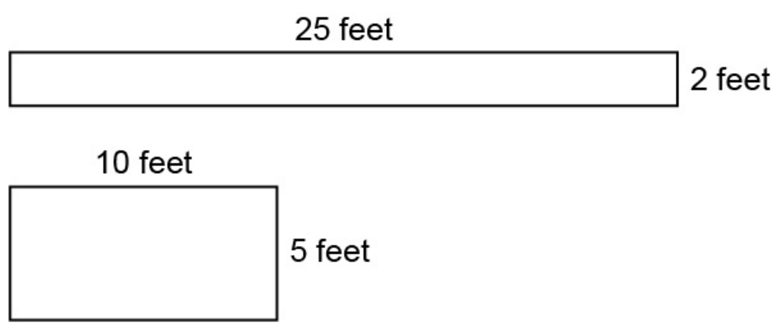
- Ask students to determine the amount of material needed to cover a rectangular area with specified side lengths. For example, if Maya builds a habitat for her rabbits that is 9 feet long and 4 feet wide, she will need 36 square feet of material to cover the floor of the habitat because $9 \times 4 = 36$. Note: The figure shown has fencing around the rabbits, and the fencing is not made of unit squares.



- Ask students to use area in making choices between two rectangles in a real-world problem. For example, a stack of 48 one-foot square tiles could be used to tile a room that is 5 feet by 9 feet but not a room that is 7 feet by 7 feet because the product of 5 and 9 is 45, and the product of 7 and 7 is 49.



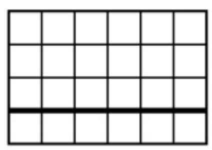
- Ask students to determine possible side lengths of rectangles that have a given area. For example, Jasper purchases a can of paint that covers an area of 50 square feet. Two possible sizes of rectangles that Jasper can paint are 2 feet by 25 feet and 5 feet by 10 feet.



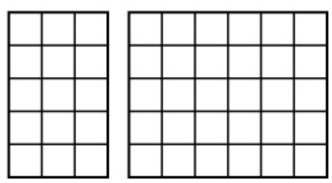
How can a tiled rectangle with whole-number side lengths be decomposed?

M.P.7. Look for and make use of structure. Illustrate that the side length of a rectangle can be rewritten as the sum of two numbers, and that when the other side is multiplied by each of those two numbers, then the sum of the products is equal to the area of the rectangle. For example, if a rectangle has side lengths of 4 units and 8 units, then 4 can be rewritten as the sum of 2 and 2. As a result, a tiled rectangle with 2 rows and 8 columns is created along with another tiled rectangle with 2 rows and 8 columns. The total number of tiles is equal to the sum of (8×2) and (8×2), which is the same as $16 + 16 = 32$. Additionally, demonstrate that all methods of decomposing a given rectangle will result in the same total area.

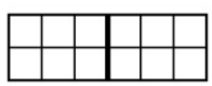
- Ask students to write an expression that shows how to calculate the area of a given rectangle. For example, give students the following diagrams. Some sample student expressions are shown next to each diagram.



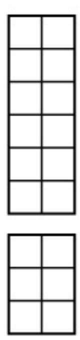
6 × (3 + 1)



(3 × 5) + (6 × 5)

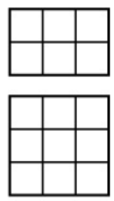


(3 + 3) × 2

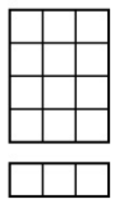


(2 × 6) + (2 × 3)

- Give students a fixed number of square tiles and ask them to create a rectangle using all the tiles. Ask students to find multiple ways of decomposing the rectangle into two smaller rectangles and then write expressions showing how to calculate the area of the decomposition. For example, give students 15 tiles. Students start by creating a 3 by 5 rectangle and then use the tiles to show ways of decomposing the rectangle. Some possible decompositions are shown along with the matching expressions.

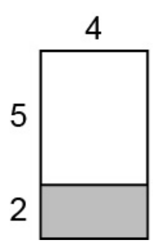


$(3 \times 2) + (3 \times 3)$



$(3 \times 4) + (3 \times 1)$

- Ask students to determine the number of unit squares in a rectangular array using two groupings, as well as using one grouping, and show that the total area is the same either way. For example, the area of the rectangle shown can be calculated by multiplying the width of 4 by the length of 7 to get 28 square units or by summing the area of the transparent portion (4×5) and the shaded portion (4×2).



Area: $4 \times (5 + 2) = (4 \times 5) + (4 \times 2)$

Key Academic Terms:

area, rectangle, tiling, array, nonoverlapping, row, column, equivalent, length, width, dimension, multiplication, product, whole number, addition, decompose

Additional Resources:

- Worksheets: Area of shapes by counting unit squares
- Activity: India's Bathroom Tiles
- Lesson: Area of rectangles: a more efficient way?
- Lesson: Grade 3 mathematics module 4, topic D, overview
- Video: Using area models and the distributive property to find area
- Activity: Introducing the distributive property

23

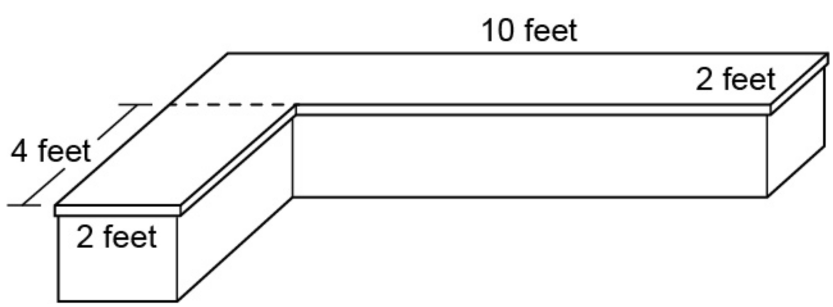
Measurement
Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
23. Decompose rectilinear figures into smaller rectangles to find the area, using concrete materials.

Guiding Questions with Connections to Mathematical Practices:

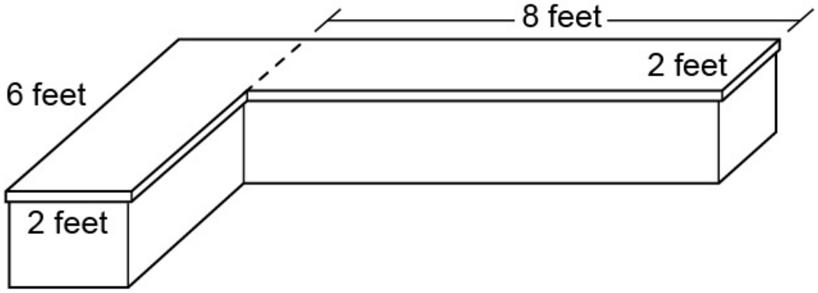
How can polygons with all right angles be decomposed to find area?

M.P.7. Look for and make use of structure. Demonstrate that rectilinear shapes can be decomposed into nonoverlapping rectangles, and that the sum of the areas of the nonoverlapping rectangles is equivalent to the area of the original rectilinear shape. For example, a U-shaped figure that is 5 units tall and 4 units wide, with an opening that is 4 units tall and 2 units wide, can be decomposed into 3 rectangles that are each 4 units by 1 unit. Additionally, the area of a rectilinear shape can be decomposed into smaller nonoverlapping rectangles in multiple ways, and the total area is independent of the way it is decomposed.

- Ask students to decompose a rectilinear shape into two or more rectangles and determine the area of the shape by finding the sum of the areas of the rectangles. For example, Jade is installing a new countertop in the shape of an L with side lengths of 2 feet, 4 feet, 8 feet, 2 feet, 10 feet, and 6 feet. Jade determines that the surface area of the countertop can be found by decomposing it into two nonoverlapping rectangles and adding the areas. So $(4 \times 2) + (10 \times 2)$ shows a total of 28 square feet.

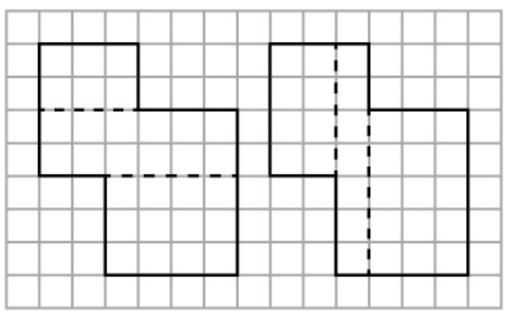


Another way to decompose the countertop is shown below.



Using the same method as above, the area is $(6 \times 2) + (8 \times 2)$ for a total of 28 square feet, the same result as decomposing in the first figure.

- Ask students to determine multiple ways to decompose rectilinear figures into smaller rectangles and show that the area of the figure is the same for all the ways it is decomposed. For example, the area of the figure shown is 30 square units and can be determined by summing rectangles with areas (2×3), (2×6), and (3×4) or by summing rectangles with areas (4×2), (7×1), and (5×3).



Key Academic Terms:

decompose, area, additive, rectilinear, equivalent, nonoverlapping

Additional Resources:

- Game: Counting squares to find area
- Activity: Finding the area of polygons

24

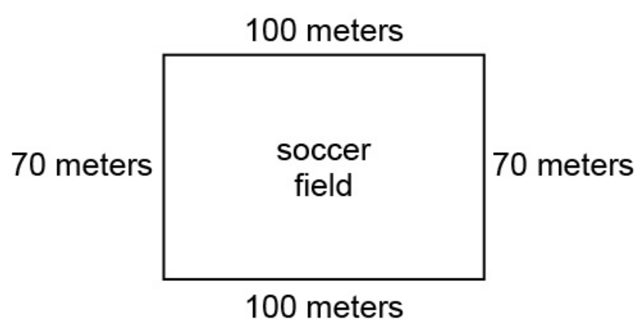
Measurement
Geometric measurement: Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.
24. Construct rectangles with the same perimeter and different areas or the same area and different perimeters.

Guiding Questions with Connections to Mathematical Practices:

What is perimeter and how is it determined?

M.P.2. Reason abstractly and quantitatively. Define perimeter as the distance around a shape, and demonstrate that perimeter can be calculated by adding all the sides of the shape together. For example, a yard that is shaped like a triangle with side lengths of 30 feet, 40 feet, and 50 feet has a perimeter of 120 feet because $30 + 40 + 50 = 120$, so 120 feet of fencing would be needed to surround the yard. Additionally, if a triangular yard has a perimeter of 120 feet and two sides are known to be 30 feet and 40 feet, it can be determined through subtraction that the third side measures 50 feet.

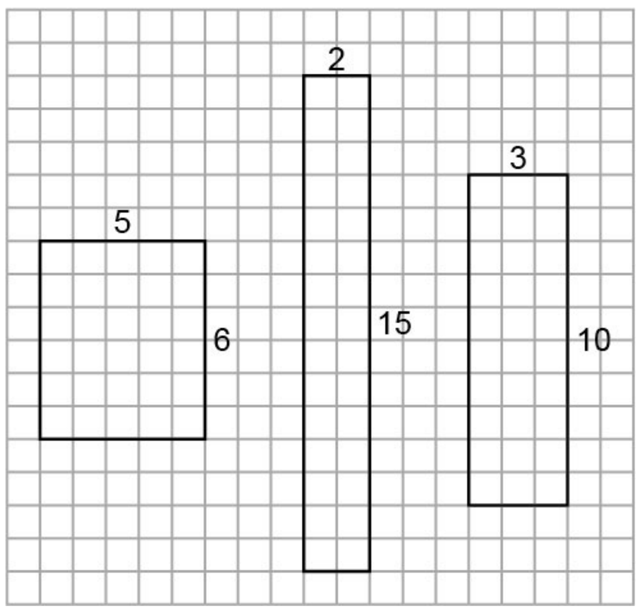
- Ask students to determine the perimeter of a figure by finding the sum of all sides. For example, if a soccer player walks around a rectangular soccer field with side lengths of 100 meters and 70 meters, the distance walked is $100 + 70 + 100 + 70 = 340$ meters.



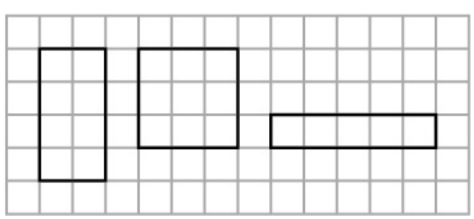
Do shapes with the same area have the same perimeter and vice versa?

M.P.2. Reason abstractly and quantitatively. Illustrate that common perimeters do not indicate common areas nor do common areas indicate common perimeters. For example, although a rectangle with side lengths of 3 units and 4 units has the same area as a rectangle with side lengths of 2 units and 6 units, the perimeter of the former is 14 units while the perimeter of the latter is 16 units. In the same way, although a rectangle with side lengths of 3 units and 5 units has the same perimeter as a rectangle with side lengths of 2 units and 6 units, the area of the former is 15 square units while the area of the latter is 12 square units. Additionally, demonstrate that among all rectangles with a common perimeter, the rectangles that are closer to square have larger areas.

- Ask students to determine the perimeter of multiple rectangles that have the same area. In the example shown, each rectangle has the same area of 30 square units, but different perimeters of 22, 34, and 26 units.



- Ask students to create multiple rectangles that have the same perimeter and different areas. For example, three rectangles, with areas 8, 9, and 5 square units (from left to right) were created with a geoboard, and each has the same perimeter of 12 units.



- Ask students to create a set of rectangles with a given area and then predict which one has the smallest perimeter. For example, using a geoboard or unit square manipulatives, create several sets of rectangles, with each set having a common area (16 square units, 25 square units, and 36 square units, respectively), concluding that in each set of rectangles, the square has the smallest perimeter.

Rectangles with areas of 16 square units

Length	Width	Perimeter
1	16	34
2	8	20
4	4	16
8	2	20
16	1	34

Rectangles with areas of 25 square units

Length	Width	Perimeter
1	25	52
5	5	20
25	1	52

Rectangles with areas of 36 square units

Length	Width	Perimeter
1	36	74
2	18	40
3	12	30
4	9	26
6	6	24
9	4	26
12	3	30
18	2	40
36	1	74

Key Academic Terms:

perimeter, polygon, area, equivalent

Additional Resources:

- Video: Calculating perimeter
- Lesson: Perimeter—measuring around

25

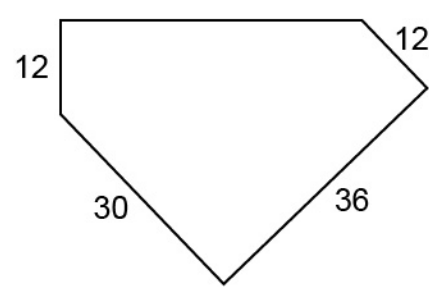
Measurement
Geometric measurement: Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.
25. Solve real-world problems involving perimeters of polygons, including finding the perimeter given the side lengths and finding an unknown side length of rectangles.

Guiding Questions with Connections to Mathematical Practices:

When can the four operations be used to find the perimeter or missing side lengths of a figure?

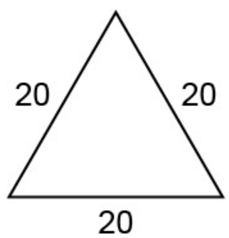
M.P.2. Reason abstractly and quantitatively. Demonstrate that if all the sides of a polygon are equal, then the perimeter can be determined by multiplying one side length by the total number of sides. For example, a square with side lengths of 4 inches has a perimeter of 16 inches because $4 \times 4 = 16$, which is equivalent to $4 + 4 + 4 + 4$. Additionally, the perimeter of a rectangle can be determined by doubling the length, doubling the width, and then adding the two products.

- Ask students to determine unknown side lengths in a figure with a given perimeter. For example, if a pentagon has a perimeter of 130 feet and given side lengths of 12 feet, 36 feet, 30 feet, and 12 feet, it can be determined that the length of the fifth side is 40 feet because $130 - 12 - 36 - 30 - 12 = 40$.

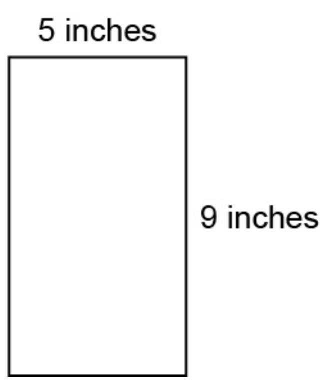


- Ask students to determine the perimeter of regular polygons by multiplying one side length by the number of sides. For example, a stop sign has side lengths of 10 inches each. The perimeter of that octagon can be calculated as $10 \times 8 = 80$ inches.

- Ask students to determine the side length of a regular polygon when given the perimeter. For example, a student has 60 centimeters of yarn to make a triangle with sides of equal length (equilateral). How long is each side? Since $60 \div 3 = 20$, each side of the triangle is 20 centimeters.



- Ask students to determine the perimeter of an oblong rectangle. For example, to calculate the perimeter of the notepad shown, students could find the sum of $(9 \times 2) + (5 \times 2)$ or students could double the sum of the two side lengths: $2 \times (9 + 5)$. Either method finds the perimeter of 28 inches.



Key Academic Terms:

perimeter, polygon, area, equivalent

Additional Resources:

- Video: Calculating perimeter
- Lesson: Perimeter—measuring around

26a

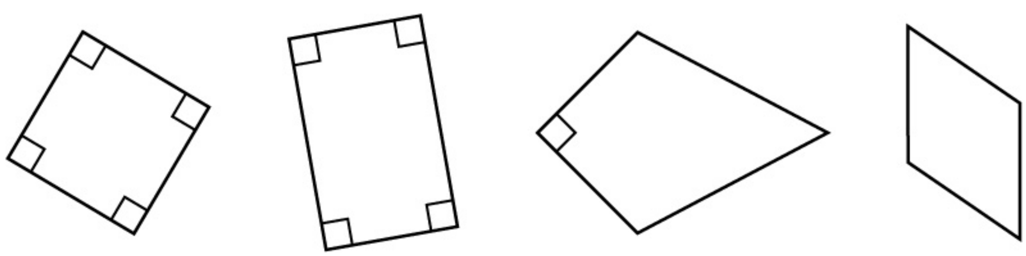
Geometry
Reason with shapes and their attributes.
26. Recognize and describe polygons (up to 8 sides), triangles, and quadrilaterals (rhombuses, rectangles, and squares) based on the number of sides and the presence or absence of square corners. a. Draw examples of quadrilaterals that are and are not rhombuses, rectangles, and squares.

Guiding Questions with Connections to Mathematical Practices:

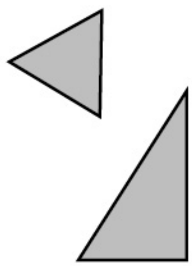
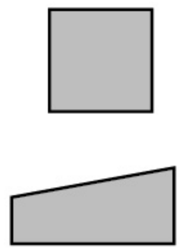
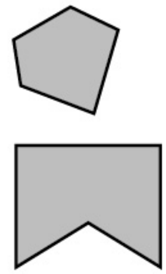
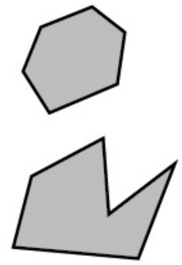
How can the attributes of a shape help to categorize the shape?

M.P.7. Look for and make use of structure. Explore the attributes of a shape to make decisions about how to categorize the shape. For example, any shape with three straight sides is a triangle, and any shape with four straight sides is a quadrilateral. Additionally, a shape may have attributes that allow it to be categorized in multiple ways.

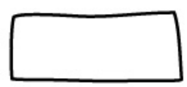
- Ask students to categorize a group of shapes based on a common attribute of those shapes. For example, the figures shown can all be categorized as quadrilaterals because they all have exactly four straight sides. Note: The right angle symbol in the corners of some of the shapes represent square corners that are right angles.



- Ask students to sort a set of polygons based on the number of sides. For example, ask students to sort the polygons shown.

Three Sides	Four Sides	Five Sides	Six Sides
			

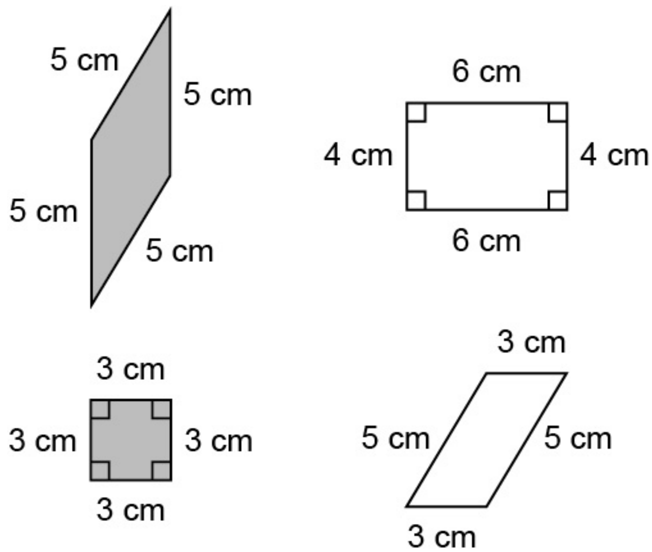
- Ask students to draw figures that have or do not have given characteristics. For example, give students the following table and ask them to draw an appropriate figure in each section. Possible student responses are included.

	Quadrilateral	Not a Quadrilateral
Has 4 right angles		
Does not have 4 right angles		

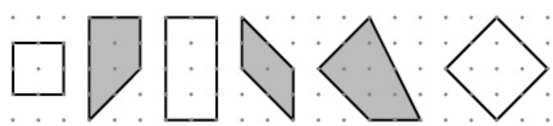
How can a shape be a quadrilateral and not belong to another subcategory of quadrilaterals (rhombus, rectangle, or square)?

M.P.3. Construct viable arguments and critique the reasoning of others. Identify the attributes that are needed to belong to the subcategories of rhombuses, rectangles, and squares, and identify when a shape does not have those attributes. For example, a quadrilateral with all four sides of different lengths will not be a rhombus, rectangle, or square. Additionally, not all quadrilaterals with two sets of equal sides are categorized as rectangles.

- Ask students to identify shapes in a collection with four sides of equal length. For example, in a set of quadrilaterals, students shaded the figures with four equal sides and categorized them as rhombuses. The remaining figures are examples of shapes that are quadrilaterals, but not rhombuses.



- Ask students to use dot paper and a pencil to construct shapes with four straight sides and identify the shapes that do not fit a particular category. For example, having drawn a variety of quadrilaterals, students shaded the figures that are not rectangles.

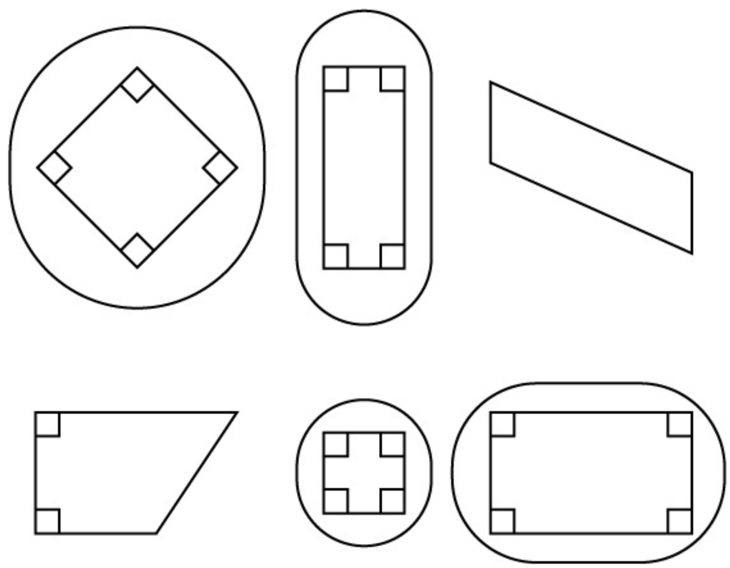


Why is a square always categorized as a rectangle, but a rectangle is not always categorized as a square?

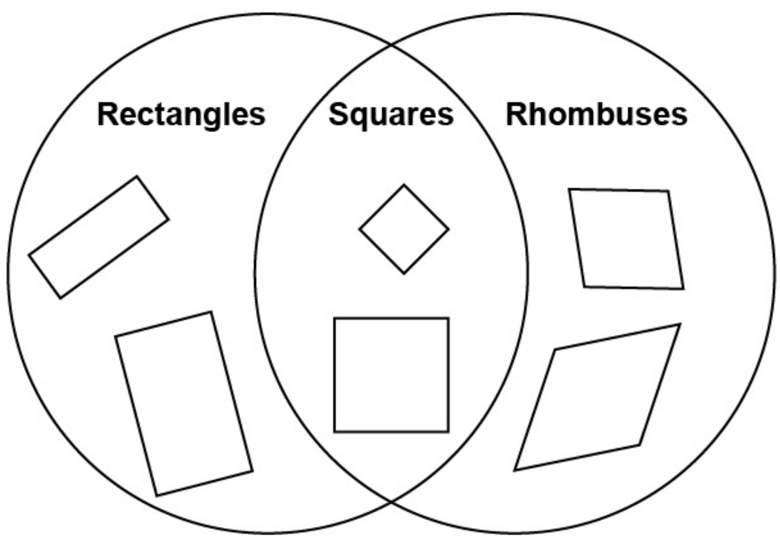
M.P.6. Attend to precision. Observe that a square has four sides of equal length and four angles of equal measure, but a rectangle only needs all four angles to be equal. As such, the definition of a rectangle is less restrictive than the definition of a square. For example, a quadrilateral with four equal angles and sides of 5 inches in length can be called both "a square" and "a rectangle," but the square categorization is more specific than the rectangle categorization. Additionally, a quadrilateral with four equal angles and sides of 5 inches in length can be called both "a square" and "a rhombus," but the square categorization is more specific than the rhombus categorization.

- Ask students to identify the defining characteristics of shapes. For example, the two attributes that are needed to determine whether a figure fits the definition of a rectangle are that it must be a quadrilateral and it must have four right angles. Any figure that has both of those characteristics is a rectangle.

- Ask students to identify all rectangles in a given set of figures. In the example shown, all rectangles are circled, including the special rectangles that are squares.



- Ask students to identify squares as uniquely fitting into the category of both rhombuses and of rectangles. In the example shown, students placed manipulatives in a Venn diagram such that only squares were in the center.



- Ask students to describe the relationship of rectangles and squares using an analogy or another example. For example, a golden retriever is a dog, just like a square is a rectangle, but the definition of a golden retriever is more restrictive than that of a dog because it is a particular kind of dog, just as the definition of a square is more restrictive than that of a rectangle because it is a particular kind of rectangle.

Key Academic Terms:

shape, attributes, category, subcategory, rhombus, rectangle, square, quadrilateral, right angle, opposite sides

Additional Resources:

- Lesson: Polygons: introduction & investigation
- Lesson: Searching for shapes in architecture
- Book: Blaisdell, M. C. B. (2009). *If you were a quadrilateral*. Mankato, MN: Picture Window Books.

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February 15, 2022