ACAP

Alabama Comprehensive Assessment Program

SUMMATIVE

Grade 4
Mathematics

Alabama Educator Instructional Supports

Alabama Course of Study Standards

DATA RECOGNITION CORPORATION

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Introduction

The *Alabama Instructional Supports: Mathematics* is a companion to the 2019 *Alabama Course of Study: Mathematics* for Grades K–12. Instructional supports are foundational tools that educators may use to help students become independent learners as they build toward mastery of the *Alabama Course of Study* content standards. **Instructional supports are designed to help educators engage their students in exploring, explaining, and expanding their understanding of the content standards.**

The content standards contained within the course of study may be accessed on the Alabama State Department of Education (ALSDE) website: https://www.alabamaachieves.org/. When examining these instructional supports, educators are reminded that content standards indicate minimum content—what all students should know and be able to do by the end of each grade level or course. Local school systems may have additional instructional or achievement expectations and may provide instructional guidelines that address content sequence, review, and remediation.

The instructional supports are organized by standard. Each standard's instructional support includes a statement of the content standard, guiding questions with connections to mathematical practices, key academic terms, and additional resources.

Content Standards

The content standards are the statements from the 2019 *Alabama Course of Study: Mathematics* that define what all students should know and be able to do at the conclusion of a given grade level or course. Content standards contain minimum required content and complete the phrase "Students will ______."

Guiding Questions with Connections to Mathematical Practices

Guiding questions are designed to create a framework for the given standards and to engage students in exploring, explaining, and expanding their understanding of the content standards provided in the 2019 *Alabama Course of Study: Mathematics*. Therefore, each guiding question is written to help educators convey important concepts within the standard. By utilizing guiding questions, educators are engaging students in investigating, analyzing, and demonstrating knowledge of the underlying concepts reflected in the standard. An emphasis is placed on the integration of the eight Student Mathematical Practices.

The Student Mathematical Practices describe the varieties of expertise that mathematics educators at all levels should seek to develop in their students. They are based on the National Council of Teachers of Mathematics process standards and the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up: Helping Children Learn Mathematics*.

The Student Mathematical Practices are the same for all grade levels and are listed below.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Each guiding question includes a representative set of sample activities and examples that can be used in the classroom. The set of activities and examples is not intended to include all the activities and examples that would be relevant to the standard.

Key Academic Terms

These academic terms are derived from the standards and are to be incorporated into instruction by the educator and used by the students.

Additional Resources

Additional resources are included that are aligned to the standard and may provide additional instructional support to help students build toward mastery of the designated standard. Please note that while every effort has been made to ensure all hyperlinks are working at the time of publication, web-based resources are impermanent and may be deleted, moved, or archived by the information owners at any time and without notice. Registration is not required to access the materials aligned to the specified standard. Some resources offer access to additional materials by asking educators to complete a registration. While the resources are publicly available, some websites may be blocked due to Internet restrictions put in place by a facility. Each facility's technology coordinator can assist educators in accessing any blocked content. Sites that use Adobe Flash may be difficult to access after December 31, 2020, unless users download additional programs that allow them to open SWF files outside their browsers.

Printing This Document

It is possible to use this entire document without printing it. However, if you would like to print this document, you do not have to print every page. First, identify the page ranges of the standards or domains that you would like to print. Then, in the print pop-up command screen, indicate which pages you would like to print.

1

Operations and Algebraic Thinking
Solve problems with whole numbers using the four operations.
1. Interpret and write equations for multiplicative comparisons.

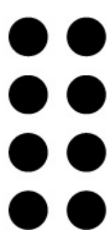
Guiding Questions with Connections to Mathematical Practices:

How are multiplicative comparisons similar to and different from equal groups and arrays?

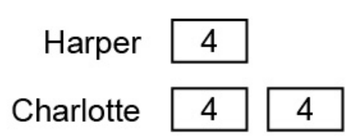
M.P.1. Make sense of problems and persevere in solving them. Describe with drawings and words the ways multiplicative comparisons are similar to and different from equal groups and arrays. For example, both problem types use multiplication to solve problems, but equal groups and arrays are focused on repeated addition (3 groups of 4 is 12), and multiplicative comparisons are interpreted as a quantity relative to other quantities as a multiple or factor (12 is 3 times as many as 4 or 4 times as many as 3). Additionally, key words can often help distinguish multiplicative comparison problems ("times as many as") from equal group/array problems ("each").

- Ask students to study a word problem with a corresponding array model and a different word problem with a corresponding multiplicative comparison model. Both models should represent the same expression. Ask students to record the multiplication expression represented by each model. Next, have a class discussion about how the models are similar and how the models are different. Guide students to see that both models are representative of the same multiplication expression, but the way the expressions are shown is different due to the context of the problem.

Jasmine has 2 plants at her house. Each plant has 4 flowers. How many flowers do the plants have in all?



Charlotte and Harper get haircuts. Charlotte gets 2 times as many inches cut off her hair as Harper. Harper gets 4 inches cut off. How many inches of hair does Charlotte have cut off?



- Ask students to determine whether various situations would best be represented by arrays (repeated addition) or multiplicative comparisons (quantities relative to other quantities). Some example situations along with possible student responses are shown.
  - James collects bugs. On Saturday, he collects 6 ladybugs, and each ladybug has 5 spots on it. How many spots, in total, are on the ladybugs James collects?

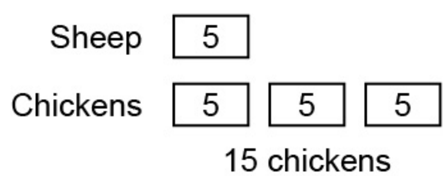
    repeated addition

  - Sadie and Parker go fishing. Parker catches 3 fish. Sadie catches 2 times as many fish as Parker. How many fish does Sadie catch?

    multiplicative comparison

- Ask students to first determine whether an array model or a multiplicative comparison model is more appropriate to use to solve a given problem and then to solve the problem using the appropriate model. Some example problems along with possible student responses are shown.
  - On a farm, there are 3 times as many chickens as there are sheep. The farm has 5 sheep. How many chickens are on the farm?

    multiplicative comparison



  - Max makes pizzas with his friends. Together they make 3 pizzas, and each pizza has 6 toppings. How many toppings, in total, are on the pizzas Max and his friends make?

    array

How can a multiplicative comparison be written as an equation and a statement?

M.P.2. Reason abstractly and quantitatively. Write equations and verbal statements to represent multiplicative comparisons. For example, read or write $24 = 6 \times 4$ as "24 is 6 times as many as 4," and write an equation for "14 is twice as much as 7" as $14 = 2 \times 7$. Additionally, since the commutative property is true of multiplication, the statement "14 is twice as much as 7" can also be represented by the equation $14 = 7 \times 2$.

- Ask students to write an equation to represent a given multiplicative comparison. Some examples along with possible student responses are shown.
  - 36 is 9 times as many as 4

    $36 = 9 \times 4$ OR $36 = 4 \times 9$
  - 64 is 8 times as many as 8

    $64 = 8 \times 8$
- Ask students to write the multiplicative comparison that a given equation represents. Some examples along with possible student responses are shown.
  - $15 = 5 \times 3$

    15 is 5 times as many as 3 OR 15 is 3 times as many as 5
  - $56 = 7 \times 8$

    56 is 7 times as many as 8 OR 56 is 8 times as many as 7

Key Academic Terms:

multiplicative comparison, times as many, product, factor, verbal statement, repeated addition, multiplication equation, multiple, array

Additional Resources:

- Article: Operations & algebraic thinking: unbound
- Activity: Thousands and millions of fourth graders

2

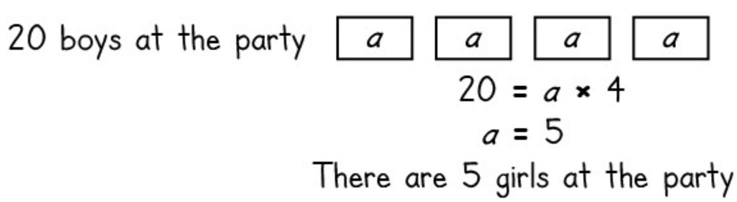
Operations and Algebraic Thinking
Solve problems with whole numbers using the four operations.
2. Solve word problems involving multiplicative comparison using drawings and write equations to represent the problem, using a symbol for the unknown number.

Guiding Questions with Connections to Mathematical Practices:

How can word problems involving multiplicative comparison be represented and solved?

M.P.4. Model with mathematics. Represent and solve a word problem with drawings and/or equations with an unknown. For example, represent the situation "There are 12 children at the playground and 3 adults. How many times as many children were at the playground as adults?" with the equation $12 = a \times 3$ and a tape diagram with a total of 12 and several groups of 3, repeating each group 4 times to solve. Additionally, the unknown number in a multiplicative comparison can either be one of the factors of the multiplication equation or the product.

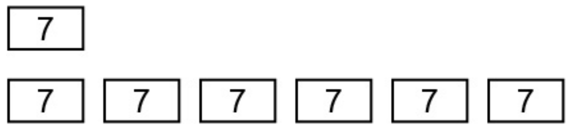
- Ask students to represent and solve multiplicative comparison word problems with a model. For example, give students the prompt "There are 20 boys at Jacob's party. There are 4 times as many boys at his party as girls. How many girls are at Jacob's party?"



- Ask students to represent and solve multiplicative comparison word problems with an equation. For example, give students the prompt "Gemma sold boxes of cookies to her family and to her neighbors for a school fundraiser. She sold 5 times as many boxes to her family as she did to her neighbors. She sold 6 boxes of cookies to her neighbors. How many boxes of cookies did she sell to her family?"

$$5 \times 6 = a, \text{ so } a = 30$$

- Ask students to use a given model and/or equation to write a multiplicative word problem that the diagram and/or equation could represent. For example, give students the model shown.



Oliver read 7 books for the school's read-a-thon. Connor read 6 times as many books as Oliver. How many books did Connor read?

As an additional example, give students the equation $8 \times a = 24$.

Mrs. Hudson has 24 biographies and 8 poetry books in her classroom library. How many times as many biographies does Mrs. Hudson have in her classroom library as poetry books?

Key Academic Terms:

product, factor, multiplication, equation, symbol, tape diagram

Additional Resources:

- Lesson: Count the times
- Lesson: Do you know the difference?
- Video: Comparison models and multiplication
- Article: Operations & algebraic thinking: unbound

3a

Operations and Algebraic Thinking
Solve problems with whole numbers using the four operations.
3. Determine and justify solutions for multi-step word problems, including problems where remainders must be interpreted. a. Write equations to show solutions for multi-step word problems with a letter standing for the unknown quantity.

Guiding Questions with Connections to Mathematical Practices:

How can the appropriate operation(s) be determined for a word problem?

M.P.1. Make sense of problems and persevere in solving them. Plan out and solve multistep word problems using mathematical operations by making sense of each problem. For example, know that a question about making the same amount of money over a span of days and then spending some of the money will involve multiplication and then subtraction. Additionally, highlighting, circling, and/or underlining key information and identifying what exactly is being asked is a helpful strategy to plan out and solve a multistep word problem.

- Ask students to study a multistep word problem and to highlight the parts of the problem that are needed in order to solve the problem. Then, ask students to underline the final question that is being asked. For example, the problem shown has the highlighting and underlining already completed.

    To earn extra money, Mylee does yardwork for her neighbors. Each neighbor pays her $2 for raking leaves and $4 for pulling weeds. Last week she raked leaves for 3 neighbors and pulled weeds for 2 neighbors. How much money did Mylee make doing yardwork last week?

    Discuss as a group what students highlighted and underlined in each problem.

- Ask students to study a series of multistep word problems and to list the operations in the correct order that they are needed to solve each problem. For example, some problems and the operations needed to solve them are shown.
    - Calvin and Riley are on the same basketball team. During their first game, Calvin scores 18 points, and Riley scores 7 fewer points than Calvin. How many points did they score in all? In this case, the operations are subtraction and then addition.
    - Grandville Elementary has 523 students in kindergarten through grade 5. There are 68 students in kindergarten, and each of the other 5 grades has the same number of students in each grade. How many students are in each of the other 5 grades? In this case, the operations are subtraction and then division.
    - Jade helps decorate for the school festival. She inflates 6 bags of red balloons. Each bag of balloons has 12 balloons. A local party supply shop donates 96 blue balloons. Right before the festival begins, 4 balloons pop. How many balloons, in total, are now at the festival? In this case, the operations are multiplication, then addition, and then subtraction.
- Ask students to solve a multistep word problem by showing each step necessary to solve it and then giving the final answer. For example, give students the problem "An elementary school with 360 students has 3 different lunch periods every day. The same number of students eat lunch during each period. The tables in the cafeteria can each seat 8 students. How many tables are needed in the cafeteria so that all students have a seat during their lunch period?" A possible student response is shown.

Step 1: $360 \div 3 = 120$

Step 2: $120 \div 8 = 15$

15 tables

How does context determine what to do with a remainder?

M.P.2. Reason abstractly and quantitatively. Know that a remainder can be interpreted in many ways, depending on the question being asked. It could be appropriate to ignore the remainder, to round up, or to partition it. For example, if children are sharing marbles equally, remaining marbles should be ignored, since they cannot be partitioned or rounded up. Additionally, if the remainder is not interpreted correctly, the entire solution could be incorrect.

- Ask students to study a series of word problems and use the context of each problem to determine which of the following should be done with the remainder: ignore the remainder, round the remainder up, or partition the remainder. For example, give students problems like those shown.

    - The two fourth-grade classes at Lakeside Elementary are going on a field trip. There are 23 students in Ms. Lee's class and 27 students in Mr. Akin's class. The students will be riding to the field trip in vans that can each fit 6 students. How many vans are needed to take all the fourth-grade students on the field trip?

      Round the remainder up.

      (Students cannot ride in a partial van.)

    - Leo's mom makes 9 vanilla cupcakes and 15 chocolate cupcakes for Leo and his 4 friends to share. She wants all the cupcakes to be eaten and for all 5 of the children to get the same number of cupcakes. How many cupcakes should Leo's mom give each child?

      Partition the remainder.

      (Children can eat parts of cupcakes that have been sliced.)

    - Abby gets $25 for her birthday. She puts $4 in her piggy bank and then brings the rest of the money to the school book fair to buy books. Each book costs $4. How many books can Abby buy?

      Ignore the remainder.

      (There are no books less than $4, so Abby cannot buy any additional books with her change.)

- Ask students to solve word problems that involve correctly interpreting a remainder. For example, give students problems like those shown.

    - Victoria makes hair bows. She has 88 inches of ribbon to make the bows; however, 5 inches of the ribbon are frayed and cannot be used. Victoria uses 8 inches of ribbon to make each bow. How many hair bows can she make with all her usable ribbon?

      10 bows

      (Ignore the remainder.)

- Grayson's dad is building him a bookcase to store all his books. Grayson has 13 nonfiction books and 33 fiction books. Each shelf of the bookcase can hold 7 books. How many shelves does Grayson's dad need to build so that the bookcase can store all of Grayson's books?

  7 shelves

  (Round the remainder up.)

- Nathan's family has 6 people including him. They order a pizza with 16 slices for dinner, but his little brother decides not to eat any because he is not feeling well. The rest of the family eats the entire pizza, and they all eat the same amount. How many slices of pizza does each other member of Nathan's family eat?

  $3\frac{1}{5}$ pizza slices

  (Partition the remainder.)

How can an equation with a letter representing the unknown be used to represent a word problem?

M.P.4. Model with mathematics. Write an equation with a letter representing the missing quantity in a word problem and solve for that quantity. For example, write the equation $(3 \times 4) + (6 \times 2) = a$ and solve for a in response to the following word problem: "Juan gives 4 apple slices to each of his 3 friends. Then he gives 6 apple slices each to his sister and brother. How many apple slices did Juan give?" Additionally, depending on the context of the problem and how the problem is set up, the letter standing for the unknown quantity can be on either side of the equals sign in the equation.

- Ask students to study a word problem and a given equation that could be used to solve the word problem. Then ask students to identify what the letter in the equation represents. For example, give students the problem "Emily gets paid for walking dogs. She gets paid $3 for every dog she walks plus any tips that her customers give her. Last week, Emily walked 7 dogs. She made a total of $26 dollars. How much money in tips did Emily receive last week?" and the equation shown.

  $$(3 \times 7) + x = \$26$$

  Students should respond that the x represents the tip money that Emily received last week.

- Ask students to write and solve an equation that could be used to represent a word problem. Each equation needs to have a letter standing for an unknown number. For example, give students the problem "Tia sold a total of 86 bags of popcorn at the school fair. The popcorn comes in three different flavors: cheddar, caramel, and butter. She sold 3 times as many bags of butter popcorn as caramel popcorn. She sold 16 bags of caramel popcorn. How many bags of cheddar popcorn did Tia sell?" A possible equation for the problem is shown, along with a solution.

$$(3 \times 16) + 16 + c = 86$$

Students should respond that $c = 22$, so Tia sold 22 bags of cheddar popcorn.

- Ask students to study a word problem and a given equation with a letter standing for an unknown quantity that could be used to solve it. Then ask students to write a different equation that could also be used to solve the problem. For example, give students the problem "There are 25 students in Mr. Hill's fourth-grade class. There are 12 students who ride the bus and 8 students who are driven to school each day. The rest of the students walk to school. How many students in Mr. Hill's fourth-grade class walk to school each day?" and the equation shown.

$$25 - (12 + 8) = w$$

A possible student response is $12 + 8 + w = 25$.

Key Academic Terms:

addition, subtraction, multiplication, division, operation, multistep problem, remainder, unknown quantity, equation, rounding, mental strategy, partition

Additional Resources:

- Article: Operations & algebraic thinking: unbound
- Worksheet: Multistep word problems
- Lesson: Two-step word problems

3b

Operations and Algebraic Thinking
Solve problems with whole numbers using the four operations.
3. Determine and justify solutions for multi-step word problems, including problems where remainders must be interpreted. b. Determine reasonableness of answers for multi-step word problems, using mental computation and estimation strategies including rounding.

Guiding Questions with Connections to Mathematical Practices:

What is the meaning of the remainder of a division problem in context?

M.P.1. Make sense of problems and persevere in solving them. Explain the meaning of the remainder both in and out of context. For example, Evan has 26 stickers and will be giving the same amount to each of his 6 friends. How many stickers will Evan have left? The key to answering this question is to recognize that the remainder is the value needed. Solving 26 ÷ 6 is 4 R2. This means Evan will have 2 stickers left. Additionally, understand that the remainder must always be less than the divisor and how this fact connects to a given context. Further, understand how to adjust the remainder to correct the situation in which the remainder is greater than or equal to the divisor.

- Ask students to identify the meaning of the remainder in various contexts. Several examples are shown.
    - A coach buys 31 water bottles and gives 2 water bottles to each of the 15 players at practice. In this case, the remainder represents the number of water bottles left over after each player has 2 water bottles.
    - A pet shop has 8 empty goldfish tanks. The pet shop owner has 35 goldfish. She puts as many goldfish as possible in each of the goldfish tanks so they each have the same number of goldfish. She puts the rest of the goldfish in a small fishbowl. In this case, the remainder represents the number of goldfish the pet shop owner puts in the small fishbowl.
    - Kristina arranges 50 chairs in a gym. She starts by making 9 identical rows of chairs. She puts as many chairs as possible in each of the rows. Then, she puts the rest of the chairs in the last row. In this case, the remainder represents the number of chairs Kristina puts in the last row.

- Ask students to explain the meaning of a quotient in context and to connect that to the meaning of the remainder. For example, Phillip cuts 23 inches of ribbon into 5-inch strips. He calculates $23 \div 5 = 4$ R3. Students should identify that the quotient of 4 indicates the number of 5-inch strips of ribbon that Phillip can cut and that the remainder of 3 indicates how close Phillip is to having another complete 5-inch strip.

- Ask students to describe, given a context, why the remainder of a division problem representing that context cannot be greater than the divisor. For example, Rafael is building model cars. He has 22 wheels. Each model car needs 4 wheels. Rafael says he can build 4 cars with 6 wheels left over. Students should identify that the remainder of 6 is greater than the divisor of 4. In this context, Rafael could use 4 of the remaining 6 wheels to build another car. This means Rafael can build 5 cars with 2 wheels left over, so $22 \div 4 = 5$ R2.

How can the reasonableness of an answer be justified?

M.P.3. Construct viable arguments and critique the reasoning of others. Identify whether an answer is reasonable without solving a problem. For example, for the equation $655 \times 9 = 4{,}585$, round 655 to 700 and solve $700 \times 9 = 6{,}300$ to realize the product is too small. Additionally, using compatible numbers that are easy to calculate is another helpful strategy to use alongside estimation when checking for the reasonableness of an answer.

- Ask students to study a multistep word problem with a given answer. Then ask students to use estimation to determine if the answer seems reasonable or not reasonable. For example, give students the problem "Kinley is participating in her school's read-a-thon. She reads every night during the first week of the read-a-thon for a total of 217 minutes. She reads for the same amount of time for each of those 7 nights. How many minutes does Kinley read each night?" and the answer of 10 minutes. Students should respond that the answer is not reasonable, because 217 is close to 210, which is divisible by 7, and $210 \div 7 = 30$, which is not very close to 10.

- Ask students to study a series of computation problems and two given answers for each problem (one from Student A and one from Student B). Then ask students to use estimation to determine which of the two students is most likely correct. Some examples along with possible student responses are shown.

- 555×4

  Student A: "I think the answer is 2,220."

  Student B: "I think the answer is 222."

  I think Student A is most likely correct because 555 rounds to 600, and $600 \times 4 = 2{,}400$, which is closer to 2,220 than it is to 222.

- $8{,}127 \div 9$

  Student A: "I think the answer is 93."

  Student B: "I think the answer is 903."

  Student B is most likely correct because 8,127 rounds to 8,100, and $8{,}100 \div 9 = 900$, which is closer to 903 than it is to 93.

- $3 \times (590 + 627)$

  Student A: "I think the answer is 3,651."

  Student B: "I think the answer is 2,397."

  Student A is most likely correct because 590 rounds to 600 and 627 also rounds to 600, and $(600 + 600) \times 3 = 3{,}600$, which is closer to 3,651 than it is to 2,397.

- $1{,}290 + 1{,}220 \div 10$

  Student A: "I think the answer is 251."

  Student B: "I think the answer is 1,412."

  Student B is most likely correct because 1,290 rounds to 1,300 and 1,220 rounds to 1,200, and $1{,}300 + (1{,}200 \div 10) = 1{,}420$, which is closer to 1,412 than it is to 251.

Key Academic Terms:

addition, subtraction, multiplication, division, operation, multistep problem, unknown quantity, equation, rounding, mental strategy, computation

Additional Resources:

- Article: Operations & algebraic thinking: unbound
- Worksheet: Multistep word problems
- Lesson: Two-step word problems

4a

Operations and Algebraic Thinking
Gain familiarity with factors and multiples.
4. For whole numbers in the range 1 to 100, find all factor pairs, identifying a number as a multiple of each of its factors. a. Determine whether a whole number in the range 1 to 100 is a multiple of a given one-digit number.

Guiding Questions with Connections to Mathematical Practices:

What patterns can be found in the factor pairs of a number or numbers?

M.P.7. Look for and make use of structure. Describe patterns found in the factor pairs of a single number or multiple numbers. For example, because 18 is a factor of 36, if you know the factors of 18, you can find the factors of 36 by noticing that 36 is double 18 and that, to find the factors of 36, you need to double one factor in the factor pairs of 18. The factor pairs of 18 are 1×18, 2×9, and 3×6. The factor pairs of 36 are 1×36, 2×18, 3×12, 4×9, and 6×6. Additionally, 6×14 is equivalent to 3×28 because the factors of 6 are 2×3, so the 3 can be used to replace the factor of 6 by using 2 times 14 to get 28.

- Ask students to identify factor pairs of a number given the set of factor pairs for a factor or multiple of that number. For example, given that the factor pairs of 100 are 1×100, 2×50, 4×25, 5×20, and 10×10, find the factor pairs for 50. Since 50 is half of 100, the factor pairs of 50 consist of all the factor pairs of 100 in which one factor can be evenly halved: 1×50, 2×25, and 5×10.

- Ask students to determine factor pairs for a set of numbers and to identify the relationship between those factor pairs. For example, consider a table created to show the factor pairs for 18, 36, and 54. Observe that the first three pairs in each list have the same first number. However, the second number of each factor pair for 36 is double the second number of each factor pair for 18, because 36 is double 18. Similarly, the second number of each factor pair for 54 is triple the second number of each factor pair for 18, because 54 is triple 18.

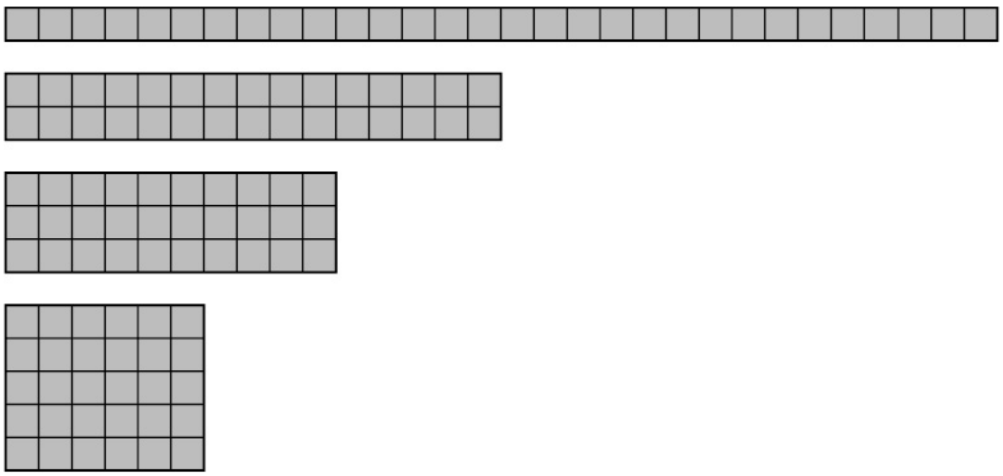
Factor Pairs for 18	Factor Pairs for 36	Factor Pairs for 54
1×18	1×36	1×54
2×9	2×18	2×27
3×6	3×12	3×18
	4×9	
	6×6	6×9

- Ask students to compare factor pairs of numbers that are relatively prime and to describe the relationship between those numbers. For example, the factor pairs for 25 are 1×25 and 5×5, while the factor pairs for 27 are 1×27 and 3×9; therefore, 25 and 27 are relatively prime as these numbers share no factors in common except for 1.

How can all factor pairs be found for a number?

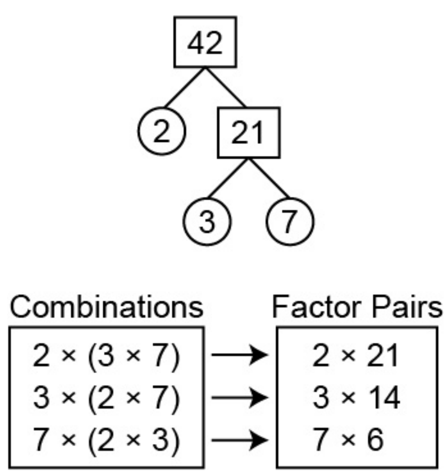
M.P.8. Look for and express regularity in repeated reasoning. Identify a strategy to know when all factor pairs of a number have been found. For example, to find the factor pairs of 12, start with the factor pair including 1 and go through each whole number until the factor pairs reverse, which occurs between 3×4 and 4×3. Then, all factor pairs of 12 (i.e., 1×12, 2×6, and 3×4) have been found. Additionally, the factor pairs of 12 can be determined by listing the prime factors of 12, which are $1 \times 2 \times 2 \times 3$, and grouping factors systematically: $1 \times (2 \times 2 \times 3)$, $(1 \times 2) \times (2 \times 3)$, and $(1 \times 2 \times 2) \times 3$.

- Ask students to determine all factor pairs for a number by creating all possible rectangles with whole-number side lengths with a given area. For example, graph paper can be used to create all the rectangles with whole-number side lengths and an area of 30 square units. Since the factor pairs for 30 are 1 × 30, 2 × 15, 3 × 10, and 5 × 6, the following rectangles can be created.



- Ask students to list all possible factor pairs for a number by listing the factor pairs in numerical order according to the first factor of each pair and then stopping the list when the first factor of a pair matches the second factor of a previous pair. For example, students may begin listing factor pairs for 24 as follows: 1 × 24, 2 × 12, 3 × 8, and 4 × 6. The next pair would be 6 × 4, but since the first factor 6 matches the second factor of the previous pair, 6 × 4 should be excluded and the list is complete.

- Ask students to determine all factor pairs of a number by grouping prime factors in a factor tree. For example, in a factor tree for 42, factor pairs can be created by using combinations of the prime factors shown in the circles.



Key Academic Terms:

multiple, factor, prime, composite, whole number, factor pair

Additional Resources:

- Lesson: Factor pairs, arrays, patterns and fun! Building factor pairs to 50
- Book: McElligott, M. (2007). *Bean thirteen*. New York, NY: G.P. Putnam's Sons Books for Young Readers. Activity
- Article: Operations and algebraic thinking: unbound

4b

Operations and Algebraic Thinking
Gain familiarity with factors and multiples.
4. For whole numbers in the range 1 to 100, find all factor pairs, identifying a number as a multiple of each of its factors. b. Determine whether a whole number in the range 1 to 100 is prime or composite.

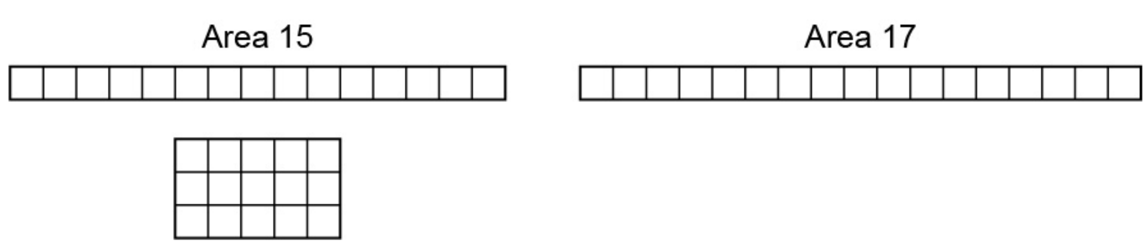
Guiding Questions with Connections to Mathematical Practices:

How can the terms "multiple," "factor," "prime," and "composite" be used to communicate precisely and solve problems?

M.P.6. Attend to precision. Define and use "multiple," "factor," "prime," and "composite" to describe numbers. For example, the factors of 6 are 1, 2, 3, and 6, or 6 is a multiple of 1, 2, 3, and 6. Because 6 has more than two factors, 6 is a composite number. Similarly, the factors of 13 are 1 and 13, or 13 is a multiple of 1 and 13. Because 13 has just two factors, 13 is a prime number. Additionally, prime numbers can be identified through systematic analysis (like the Sieve of Eratosthenes) in which multiples of prime numbers are tagged as composite, and the remaining untagged numbers are prime.

- Ask students to explain why a number is prime or composite. For example, 41 is prime because its only factors are 1 and 41, while 40 is composite because its factors are 1, 2, 4, 5, 8, 10, 20, and 40.

- Ask students to determine whether a number is prime by creating rectangles with a specified area. For example, 15 is composite (not prime) because there are two unique rectangles with whole-number side lengths that can be created with an area of 15, but 17 is prime because there is only one unique rectangle that can be created with an area of 17.



- Ask students to use a hundred grid to decide whether the numbers from 2 to 100 are prime or composite. For example, a common method is the Sieve of Eratosthenes, where first the multiples of 2 are filled in, then the multiples of 3, and so on. Eventually only the prime numbers will remain unshaded. The hundred grid below shows the results of this method, where the unshaded and circled numbers are prime.

~~1~~	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Key Academic Terms:

multiple, factor, prime, composite, whole number, factor pair

Additional Resources:

- Lesson: Factor pairs, arrays, patterns and fun! Building factor pairs to 50
- Book: McElligott, M. (2007). *Bean thirteen*. New York, NY: G.P. Putnam's Sons Books for Young Readers. Activity
- Article: Operations and algebraic thinking: unbound

5

Operations and Algebraic Thinking
Generate and analyze patterns.
5. Generate and analyze a number or shape pattern that follows a given rule.

Guiding Questions with Connections to Mathematical Practices:

How can a pattern be created from a rule?

M.P.7. Look for and make use of structure. Generate a number or shape pattern when given a rule. For example, use the rule "Multiply by 2" and the starting number 1 to generate the sequence 1, 2, 4, 8, 16 . . . and observe that the rule continues. Additionally, use the rule "start with one column of three blocks and add another column in each step" to generate a sequence of rectangles each having a height of three blocks and a length of successive integers starting with 1.

- Ask students to create a list of numbers when given a rule. For example, given the rule "Double and then add 1" and the starting number of 2, students create the list 2, 5, 11, 23, 47 . . ..

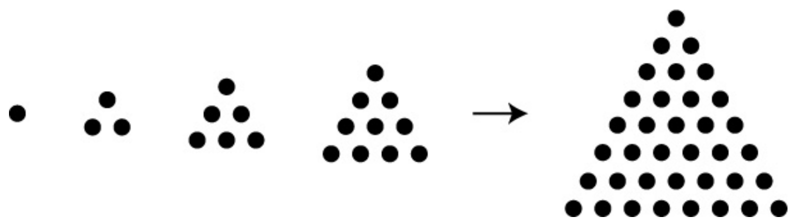
- Ask students to create a pattern from a description that models an increase at a constant rate. For example, a cyclist traveling at a constant speed of 15 miles per hour will continue to add to her distance traveled each hour that she rides, and a table can be used to show this pattern numerically.

Hours of Cycling	Miles Traveled
0	0
1	15
2	30
3	45
4	60

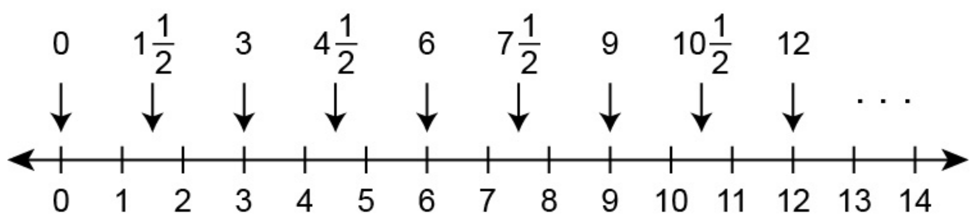
How can the properties of a rule or pattern be used to extend a pattern?

M.P.8. Look for and express regularity in repeated reasoning. Analyze a rule or pattern of numbers or shapes to extend the pattern. For example, given a sequence of square designs where each design has two more squares than the previous and the first design has 1 square, reason about how the squares are organized to determine that the 10th design has 19 squares. Additionally, given a sequence of numbers starting at 1, where all the following numbers are multiples of 4, the 10th term can be determined by multiplying 10×4.

- Ask students to identify a rule from a pattern. For example, given a sequence of triangular figures, the rule is “add a bottom row to the triangle so that all three sides have an equal number of dots,” so the 8th term in the sequence can be determined by creating an equilateral triangle with 8 dots on each side.



- Ask students to use tools to extend a pattern. For example, using a number line, students can extend the pattern starting with 0, $1\frac{1}{2}$, 3, $4\frac{1}{2}$, 6 . . ..



What are some features of a given pattern that are not explicit in the pattern's rule?

M.P.3. Construct viable arguments and critique the reasoning of others. Analyze a pattern to identify features of the pattern that are not explicit in the rule. For example, using the pattern 2, 6, 18, 54 . . . from the rule "Multiply by 3" and the starting number 2, identify that all the numbers are even and are also multiples of 3, except for 2. Additionally, using the pattern 1, 4, 9, 16, 25 . . . from the rule "Multiply the term number by itself," the differences between each two consecutive terms are odd numbers that form a sequence of their own.

- Ask students to determine alternate rules for patterns, noting features that are not explicit in the rule. For example, given the rule "starting with 1, the next term is the sum of all the preceding terms," the sequence 1, 1, 2, 4, 8, 16, 32 . . . can be created. An alternate rule could be "Multiply by 2," however this rule does not apply to the first two terms.

Key Academic Terms:

generate, rule, pattern, sequence, term, continue, identify, explicit

Additional Resources:

- Activity: Guess my rule: The Function Machine game
- Lesson: What is the function?
- Worksheets: 4.OA.C.5
- Lesson: Rules, number patterns, and tables

6

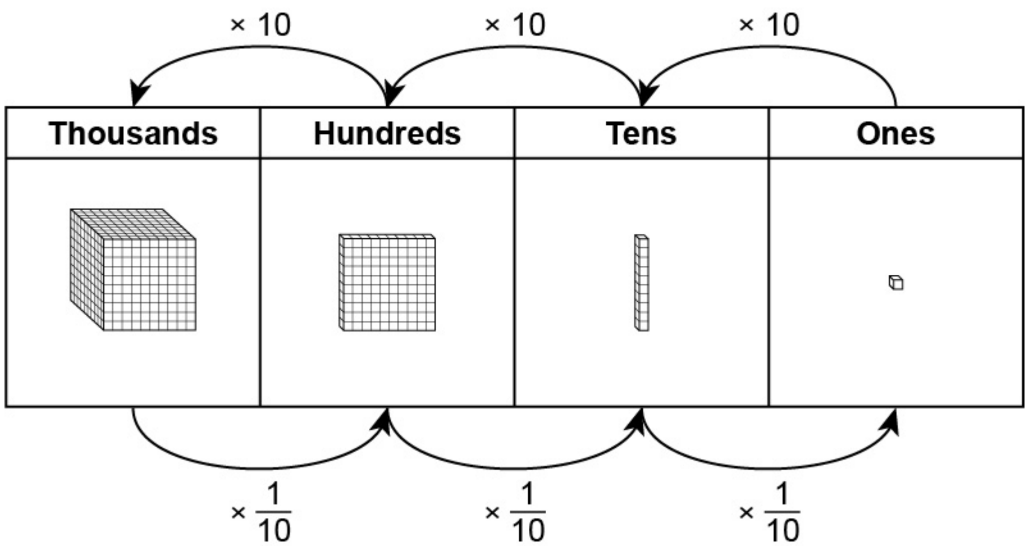
Operations with Numbers: Base Ten
Generalize place value understanding for multi-digit whole numbers.
6. Using models and quantitative reasoning, explain that in a multi-digit whole number, a digit in any place represents ten times what it represents in the place to its right.

Guiding Questions with Connections to Mathematical Practices:

How does a digit in any place compare to the same digit in the place to the right?

M.P.2. Reason abstractly and quantitatively. Know that each place value represents a different-sized unit and that, when comparing the place values of two digits that are next to each other, the place value of the digit on the right is $\frac{1}{10}$ the place value of the digit on the left. The place value of the digit on the left is 10 times the place value of the digit on the right. For example, when multiplying 813 by 10, the value of each digit is multiplied by 10, shifting into one greater place value. The 8 in the hundreds place becomes 8 thousands, 1 ten becomes 1 hundred, 3 ones become 3 tens, and there are 0 ones. Additionally, observe that the digits in the number do not affect the relationship between the place values of the numbers.

- Ask students to use a model to compare adjacent digits in a multi-digit whole number. For example, model 1 thousand, 1 hundred, 1 ten, and 1 one as shown.



Students should notice that each value is multiplied by 10 going from right to left, and by $\frac{1}{10}$ when going from left to right.

- Ask students to identify two digits in a given number such that one digit has a value that is $\frac{1}{10}$ the value of the other digit. For example, given the number 3,886, identify that the 8 in the tens place has a value that is $\frac{1}{10}$ the value of the 8 in the hundreds place because 80 is $\frac{1}{10}$ of 800.

- Ask students to identify the corresponding value of each digit when a multi-digit whole number is multiplied by 10. For example, given the multiplication problem 273×10, identify that in the factor 273, the 2 represents 2 hundreds, the 7 represents 7 tens, and the 3 represents 3 ones. When multiplying 273 by 10, the value of each digit is multiplied by 10. Therefore, in the product, the value of each digit is multiplied by 10.
    - 2 hundreds times 10 is 2 thousands
    - 7 tens times 10 is 7 hundreds
    - 3 ones times 10 is 3 tens

  Therefore, the product is $273 \times 10 = 2{,}730$.

Key Academic Terms:

place value, division, multiplication, multi-digit, times, digit

Additional Resources:

- Article: Activities for comparing numbers in the hundreds
- Activity: Place value game

7

Operations with Numbers: Base Ten
Generalize place value understanding for multi-digit whole numbers.
7. Read and write multi-digit whole numbers using standard form, word form, and expanded form.

Guiding Questions with Connections to Mathematical Practices:

How are commas used to indicate place values for multi-digit whole numbers?

M.P.7. Look for and make use of structure. Explain the structure of the base-ten system to help read and write whole numbers in a variety of ways. For example, when reading 328,908, notice that the first group of three numbers (called a "period") is read as hundreds, tens, and then ones, followed by thousands (the base-thousand unit). The second group of three numbers is also read as hundreds, tens, and ones, giving the entire phrase the following value: three hundred twenty-eight thousand nine hundred eight. Additionally, note that the periods are indicated with commas and have at most 3 place values. Further, identify the repeated structure within each period of hundreds, tens, and ones.

- Ask students to note that within each period, the place values repeat: hundreds, tens, and ones, each corresponding to the appropriate period. For example, given the number 295,386,417, note that the 2, 3, and 4 each represent hundreds of the amount corresponding to their respective period.

  295,386,417

  - The 2 represents 2 hundred million.
  - The 3 represents 3 hundred thousand.
  - The 4 represents 4 hundred.

  Likewise, the 9, 8 and 1 represent 90 million (where 90 is equivalent to 9 tens), 80 thousand (where 80 is equivalent to 8 tens), and 10, respectively. Finally, the 5, 6, and 7 represent 5 million, 6 thousand, and 7, respectively.

- Ask students to read numbers ending in multiple zeros in which all the nonzero digits are in the same period. For example, when reading 425,000,000, observe that the 425 is in the period relating to millions, so the number should be read as "four hundred twenty-five million."

- Ask students to read numbers as three-digit numbers followed by the appropriate amount for the period. For example, 942,165,673 is read as "nine hundred forty-two million, one hundred sixty-five thousand, six-hundred seventy-three." Identify the three-digit numbers "nine hundred forty-two," "one hundred sixty-five," and "six-hundred seventy-three" within the name of the number.

How are number names related to the expanded form of a number?

M.P.7. Look for and make use of structure. Explain how knowing the number name can be used to write the expanded form of the number by identifying what power of 10 should be multiplied by each digit to generate the value of the digit in expanded form. For example, 275,824 can be divided into 275 in the thousands period and 824 in the ones period. The 275 in the thousands period can be expanded to $(200 + 70 + 5) \times 1{,}000$, and 824 can be expanded to $800 + 20 + 4$. By using properties of operations, the thousands period can be simplified to become the expanded form of $200 \times 1{,}000 + 70 \times 1{,}000 + 5 \times 1{,}000$, or $200{,}000 + 70{,}000 + 5{,}000$. Combining this with the expansion of 824 results in $200{,}000 + 70{,}000 + 5{,}000 + 800 + 20 + 4$. Additionally, know that 0s in the standard form of a number are omitted when writing the number in expanded form.

- Ask students to identify the place value associated with digits in a given number in standard form. For example, given 635,824, identify the place values shown using a place value table.

Place Value

Thousands			,	Ones		
Hundred Thousands	Ten Thousands	One Thousands	,	Hundreds	Tens	Ones
6	3	5	,	8	2	4

- Ask students to write numbers within a period in expanded form and multiply each digit by the quantity associated with the period. For example, given 2,183,576, consider the millions period (comprised only of the 2), the thousands period (comprised of 183), and the ones period (comprised of 576).

  $$2{,}183{,}576$$

  - The 2 can be represented as $2 \times 1{,}000{,}000$.
  - The 183 can be represented as $183 \times 1{,}000$, which is equivalent to $(100 + 80 + 3) \times 1{,}000$.
  - The 576 can be represented as 576×1, which is equivalent to $500 + 70 + 6$.
- Ask students to write the expanded form of a given number. For example, given 4,507,623, write the expanded form of the number as shown.

  $$(4 \times 1{,}000{,}000) + (5 \times 100{,}000) + (7 \times 1{,}000) + (6 \times 100) + (2 \times 10) + 3$$

  or

  $$4{,}000{,}000 + 500{,}000 + 7{,}000 + 600 + 20 + 3$$

  The original number can also be decomposed into periods and then written in expanded form. The number 4,507,623 can also be written as the following.

  $$(4 \times 1{,}000{,}000) + (507 \times 1{,}000) + (623)$$

  Expanding the 507 and 623 gives $(4 \times 1{,}000{,}000) + (500 + 7) \times 1{,}000 + (600 + 20 + 3)$. This can be rewritten, using properties of multiplication, as the expanded form given previously.

Key Academic Terms:

base-ten numerals, expanded form, number name, comma, place value, properties of operations, period, standard form, word form, decompose

Additional Resources:

- Article: Activities for comparing numbers in the hundreds
- Activity: Place value game
- Video: Expanded form and place value, Maths Working Model
- Lesson: My digit is bigger than your digit! Comparing multi-digit numbers

8

Operations with Numbers: Base Ten
Generalize place value understanding for multi-digit whole numbers.
8. Use place value understanding to compare two multi-digit numbers using >, =, and < symbols.

Guiding Questions with Connections to Mathematical Practices:

How can multi-digit numbers be compared?

M.P.8. Look for and express regularity in repeated reasoning. Explain the connection between comparisons of two numbers of any value to comparisons made in previous learning. For example, when comparing 1,428 and 2,093, know that, starting with the highest place value, $1{,}428 < 2{,}093$ because 1,000 is less than 2,000. Additionally, observe that if the digits in the highest place value of each number are equal, the digits in the next-highest place value should be compared, and so on. Further, note that if one of the numbers being compared has a first digit in a higher place value, that number must be greater.

- Ask students to compare two numbers with an equal number of digits in which the leading digits are different and record their results using the appropriate inequality sign. For example, ask students to compare 78,324 and 81,965. Both numbers have digits in the ten-thousands place. Because $70{,}000 < 80{,}000$, the inequality $78{,}324 < 81{,}965$ compares the two numbers.

- Ask students to compare two numbers with an equal number of digits in which the leading digits are the same and record their results using the appropriate inequality sign. For example, given the numbers 23,146 and 23,089, a comparison of the leading digits (in the ten-thousands place) shows they are both 2. Moving to the thousands place, a comparison shows they are both 3. Moving to the hundreds place, a comparison shows that 1 hundred is greater than 0 hundreds. Therefore, $23{,}146 > 23{,}089$.

Place Value

Ten Thousands	One Thousands	,	Hundreds	Tens	Ones
2	3	,	1	4	6
2	3	,	0	8	9

- Ask students to compare two numbers with different numbers of digits and record their results using the appropriate inequality sign. For example, given the numbers 145,918 and 78,599, because the first digit in 145,918 is in the hundred-thousands place and 78,599 does not have a digit in the hundred-thousands place, 145,918 must be greater than 78,599. This comparison can be written as $145,918 > 78,599$.
- Ask students to compare multi-digit whole numbers using >, =, or < symbols. For example, give students a table of numbers. Have them write the appropriate symbol between each pair of numbers as shown.

Comparisons

999,999	>	99,999
123	<	1,234
86,000	=	86,000
103,488	<	130,488

Key Academic Terms:

base-ten numerals, expanded form, number name, comma, place value, properties of operations, period, standard form, compare, digit

Additional Resources:

- Article: Activities for comparing numbers in the hundreds
- Activity: Place value game
- Video: Expanded form and place value, Maths Working Model
- Lesson: My digit is bigger than your digit! Comparing multi-digit numbers

9

Operations with Numbers: Base Ten
Generalize place value understanding for multi-digit whole numbers.
9. Round multi-digit whole numbers to any place using place value understanding.

Guiding Questions with Connections to Mathematical Practices:

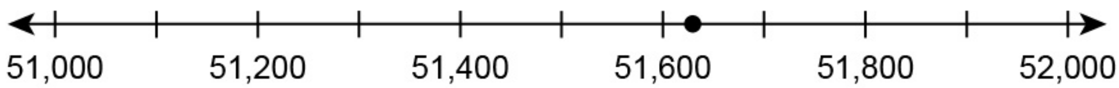
How is rounding past the hundreds place related to rounding to the tens and hundreds?

M.P.8. Look for and express regularity in repeated reasoning. Extend rounding to more place values by using similar methods to previous learning about rounding. For example, use a number line strategy to round to tens and hundreds and then use the same strategy to round to thousands. Additionally, identify the place value which determines how a given number is to be rounded, based on the place it is being rounded to.

- Ask students to identify the digit that determines how a number should be rounded. For example, when asked to round 26,854 to the nearest thousand, verify that the place value that determines how the number should be rounded is not the thousands place, but the next smallest place—in this case, the hundreds place. Similarly, when asked to round 532,168 to the nearest hundred thousand, the digit in the ten-thousands place determines how the number should be rounded.

- Ask students to round a given number to a given place value. For example, when asked to round 742,581 to the nearest hundred-thousand, verify that 742,581 should be rounded down because the digit in the next smallest place after the hundred-thousands place—in this case, the 4 in the ten-thousands place—is less than 5. Therefore, 742,581 should be rounded down to 700,000.

- Ask students to round a given number to a given place based on the location of the number on a number line. For example, when asked to round 51,627 to the nearest thousand, first create a number line containing the multiples of 1,000 just before and after 51,627—in this case, 51,000 and 52,000. The relevant part of the number line with a point plotted at 51,627 is shown.

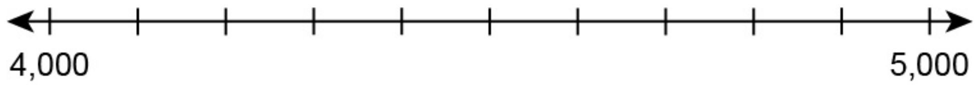


Because the distance from 51,627 to 52,000 is less than 4 hundreds and the distance from 51,627 to 51,000 is more than 6 hundreds, the value of 51,627 when rounded to the nearest 1,000 is 52,000.

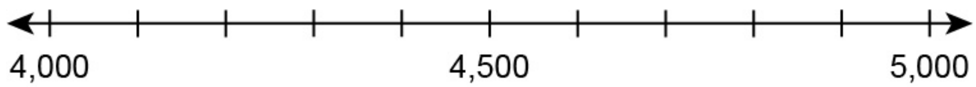
What makes the digit "5" in the hundreds place significant when rounding to the nearest thousand?

M.P.7. Look for and make use of structure. Identify that the digit 5 in the hundreds place is significant because it equals 500, which is the halfway point between two thousands on a number line. For example, on a number line from 3,000 to 4,000, the interval between 3,000 and 4,000 is divided into ten equal-sized sections that are marked with the numbers 3,100 to 3,900 counting by one hundreds. The number 3,500 is the same distance from 3,000 as it is from 4,000 on the number line, so it represents the halfway point between 3,000 and 4,000. To the left of 3,500, all the values are closer to 3,000, and to the right of 3,500, all of the values are closer to 4,000. The value of 3,500 will round up to 4,000 because half of the values round to 3,000 and half of the values round to 4,000. The value 3,500 is the halfway point, so the digit 5 in the hundreds place is significant when rounding to the thousands. The same is true when rounding to any place value; the digit 5 is significant when rounding to a place value that is 10 times the value of the placement of the 5. For example, a 5 in the tens place is important when rounding to the hundreds place. Additionally, identify that a 5 in the hundreds place is only significant when rounding to the nearest thousand, even if the number being rounded extends to the ten-thousands place or beyond.

- Ask students to identify the halfway point between two consecutive thousands on a number line. For example, give students the number line shown.



Because there are 10 intervals on the number line between 4,000 and 5,000, the halfway point between 4,000 and 5,000 is the point below which there are 5 intervals and above which there are 5 intervals. That point is 4,500 and can be located by counting up by 100s from 4,000 or down by 100s from 5,000. The location of 4,500 is shown on the number line.



- Ask students to round numbers with a 5 in the hundreds place to the nearest thousand. For example, to round 8,569 to the nearest thousand, identify the hundreds place as the relevant place. Also, identify that 8,569 is between 8,000 and 9,000. Because there is a 5 in the hundreds place, 8,569 is rounded up to the nearest thousand, which is 9,000. As an additional example, to round 24,513 to the nearest thousand, again identify the hundreds place as the relevant place (because the rounding is to the nearest thousand, even though the number 24,513 extends to the ten-thousands place). Because there is a 5 in the hundreds place, 24,513 is rounded up to the nearest thousand, which is 25,000.

Key Academic Terms:

round, place value, ones, tens, hundreds, thousands, ten-thousands, hundred-thousands, millions, about, approximately, halfway point, interval, digit

Additional Resources:

- Article: Teaching rounding so students actually understand
- Video: Teaching rounding with MAB

10

Operations with Numbers: Base Ten
Use place value understanding and properties of operations to perform multi-digit arithmetic with whole numbers.
10. Use place value strategies to fluently add and subtract multi-digit whole numbers and connect strategies to the standard algorithm.

Guiding Questions with Connections to Mathematical Practices:

Why does the standard algorithm for addition and subtraction work?

M.P.3. Construct viable arguments and critique the reasoning of others. Use the standard algorithm for addition and subtraction to solve problems involving multi-digit whole numbers with regrouping and explain why the standard algorithm works. For example, while subtracting with the standard algorithm, it is important to remember that the following concept can be applied when regrouping: one ten is equivalent to a bundle of ten ones, and one hundred is equivalent to a bundle of ten tens, and so on. Additionally, observe that when subtracting, each digit in a specific place value in the subtrahend must be less than or equal to the same place value digit in the minuend, and regrouping can be used to make each place value of the minuend greater.

- Ask students to add two numbers using the standard algorithm, including situations in which regrouping is required. For example, to add $428 + 356$, first add the digits in the ones place. Since $8 + 6 = 14$, 10 of the 14 ones should be bundled to make an additional ten, leaving just 4 ones. Then, add the digits in the tens place, remembering to include the ten carried from the addition in the ones place. Therefore, the number of tens in the sum is $1 + 2 + 5 = 8$ tens ($10 + 20 + 50 = 80$). Finally, add the digits in the hundreds place, which gives $4 + 3 = 7$ hundreds ($400 + 300 = 700$). Therefore, $428 + 356 = 784$.

- Ask students to subtract two numbers using the standard algorithm in situations where no regrouping is required. For example, to subtract $974 - 312$, first subtract the digits in the ones place, so the ones digit of the difference is $4 - 2 = 2$ ones. Then, subtract the digits in the tens place, so the tens digit of the difference is $7 - 1 = 6$ tens. Finally, subtract the digits in the hundreds place, so the hundreds digit of the difference is $9 - 3 = 6$ hundreds. Therefore, $974 - 312 = 662$.

- Ask students to subtract two numbers using the standard algorithm in situations where regrouping is required. For example, to subtract 827 – 354, first subtract the digits in the ones place. The ones digit of the difference is 7 – 4 = 3 ones. Then, note that the algorithm would require subtracting 50 from 20 to find the tens digit of the difference. Since 50 is greater than 20, regroup one of the 8 hundreds from 827 into 10 tens, since 10 tens equals 100. Now, the number 827 can be thought of as having 7 hundreds, 12 tens, and 7 ones. To find the difference in tens, subtract 120 – 50 to get 70. Finally, in order to find the hundreds digit of the difference, recall that 827 now has only 7 hundreds because 1 of the hundreds was bundled into 10 tens. The hundreds difference, then, is 700 – 400 = 300. Therefore, 827 – 354 = 473.

Step 1: $$\begin{array}{r} 827 \\ -\,354 \\ \hline 3 \end{array}$$

Step 2: $$\begin{array}{r} {\scriptstyle 7\,12} \\ \cancel{8}\cancel{2}7 \\ -\,354 \\ \hline 73 \end{array}$$

Step 3: $$\begin{array}{r} {\scriptstyle 7\,12} \\ \cancel{8}\cancel{2}7 \\ -\,354 \\ \hline 473 \end{array}$$

How does the standard algorithm connect with other addition and subtraction strategies?

M.P.1. Make sense of problems and persevere in solving them. Connect the standard algorithm and other addition strategies to make sense of the standard algorithm. For example, when solving 1,326 + 46, students should know that placing the 1 above the tens place means a ten is being added. Additionally, when solving 826 – 75, students should know that crossing out the 8 and writing a 7, along with changing the 2 to a 12, means that a single hundred was regrouped into 10 tens.

- Ask students to add numbers using the standard algorithm and explain what the notation used in the standard algorithm represents. For example, ask students to solve $975 + 248$ with the standard algorithm. Example work is shown.

$$
\begin{array}{l}
\ \ 100 \quad\ 10 \\
\ \ 900 + 70 + 5 \\
\ \ \underline{200 + 40 + 8} \\
1200 + 20 + 3 = 1223
\end{array}
\qquad\qquad
\begin{array}{r}
{\scriptstyle 1\,1}\ \\
975 \\
+\ 248 \\
\hline
1223
\end{array}
$$

Then ask students to discuss what the 1s written above the tens column and the hundreds column represent. Students should know that the 1 written over the tens column represents an additional ten that is being added and the additional ten comes from the fact that $5 + 8 = 13$, which consists of 1 ten and 3 ones. Likewise, students should know that the 1 written over the hundreds column shows that an additional hundred is being added and the additional hundred comes from the fact that $1 + 7 + 4 = 12$ tens, which consists of 1 hundred and 2 tens.

- Ask students to subtract numbers using the standard algorithm and explain what the notation used in the standard algorithm represents. For example, ask students to solve 341 – 178 with the standard algorithm. Example work is shown.

$$
\begin{array}{l}
\qquad\quad 130 \\
\ \ 200 \quad \cancel{30} \quad 11 \\
\ \ \cancel{300} + \cancel{40} + \cancel{1} \\
\underline{-\ 100 + 70 + 8} \\
\ \ 100 + 60 + 3 = 163
\end{array}
\qquad\qquad
\begin{array}{r}
{\scriptstyle 13}\ \ \\
{\scriptstyle 2\,\cancel{3}\,11} \\
\cancel{3}\cancel{4}\cancel{1} \\
-\ 178 \\
\hline
163
\end{array}
$$

Then ask students to discuss what the 11, 3, 13, and 2 each represent. Students should know the following.

  - The 11 represents the number of ones in the minuend after decomposing one of the tens into 10 ones.
  - The 3 represents the number of tens in the minuend after decomposing one of the tens into 10 ones
  - The 13 represents the number of tens in the minuend after decomposing one of the tens into 10 ones and decomposing one of the hundreds into 10 tens.
  - The 2 represents the number of hundreds in the minuend after decomposing one of the hundreds into 10 tens.

Where are numbers decomposed and composed in the standard algorithm?

M.P.1. Make sense of problems and persevere in solving them. Identify when and where numbers are decomposed in the standard algorithm to gain insight into the algorithm. For example, when adding $398 + 84$, adding $8 + 4$ makes 12. The 12 is decomposed into 10 and 2, and the 2 ones are written in the ones place and the 1 ten is added to the tens place. Then, 1 ten + 9 tens + 8 tens equals 18 tens. The 18 tens are decomposed as a 10 and an 8. The 8 tens are written in the tens place and the 10 tens are added to the hundreds place as 1 hundred. Finally, 1 hundred + 3 hundreds equals 4 hundreds. Additionally, identify regrouping in the subtraction algorithm as a decomposition of the larger place value and a composition of an additional 10 units of the smaller place value.

- Ask students to identify place values in given addition problems for which composing and decomposing is required. For example, given the sum $495 + 807$, students should observe that, because the sum of the ones digits will exceed 9, the sum of the ones digits will need to be decomposed. The sum of the tens digits will also exceed 9 due to the additional ten from the sum of the ones digits, so the sum of the tens digits will also need to be decomposed. Finally, since the sum of the hundreds digits will also exceed 9, the sum of the hundreds digits will also need to be decomposed.

- Ask students to identify place values in given subtraction problems for which composing and decomposing is required. For example, given the difference $2{,}148 - 1{,}157$, the ones place will not require any composing or decomposing since 8 is greater than 7. However, the tens place will require the decomposing of the 1 in 2,148 from 1 hundred into 0 hundreds and 10 tens. The 10 tens will be composed with the existing 4 tens to make 14 tens. Then, the hundreds place will require the decomposing of the 2 in 2,148 from 2 thousands into 1 thousand and 10 hundreds. Finally, the thousands place will not require any composing or decomposing since both thousands digits are 1 once the 2 has been decomposed.

Key Academic Terms:

addition, subtraction, standard algorithm, place value, decompose, compose, regroup, digit, column

Additional Resources:

- Lesson: Three-digit addition with trading

- Lesson: Switching to subtraction

11a

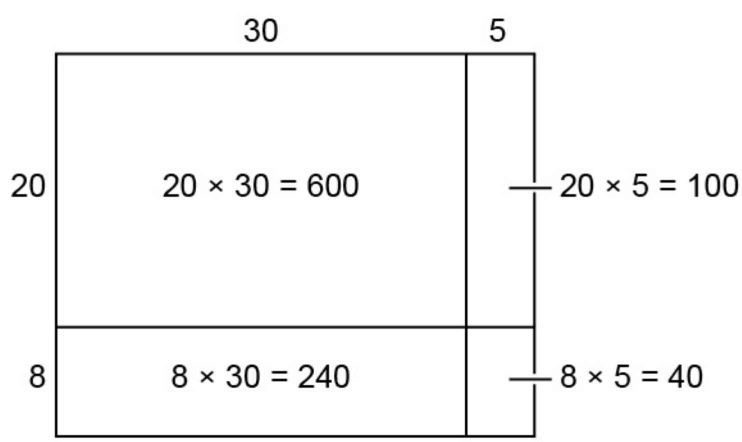
Operations with Numbers: Base Ten
Use place value understanding and properties of operations to perform multi-digit arithmetic with whole numbers.
11. Find the product of two factors (up to four digits by a one-digit number and two two-digit numbers), using strategies based on place value and the properties of operations. a. Illustrate and explain the product of two factors using equations, rectangular arrays, and area models.

Guiding Questions with Connections to Mathematical Practices:

How are area models and arrays related to equations and multiplication strategies?

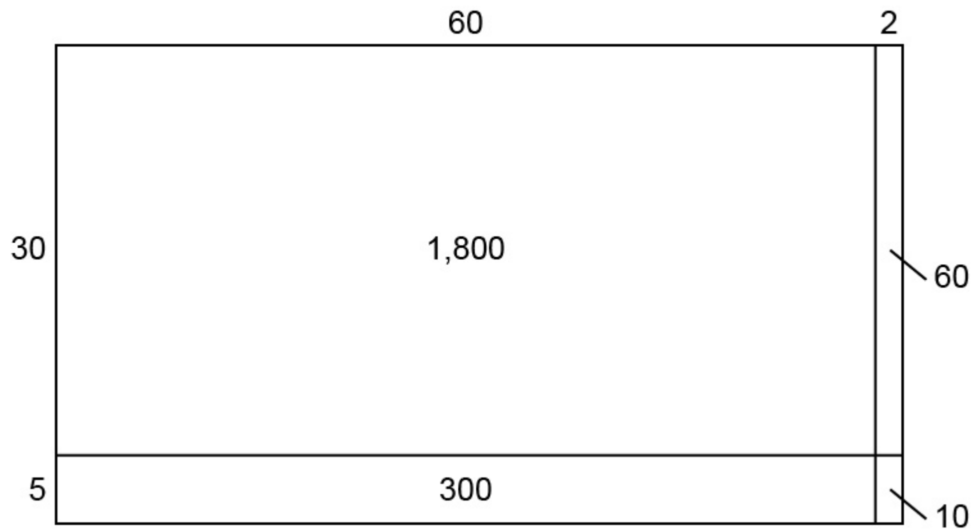
M.P.4. Model with mathematics. Connect area models and arrays to equations and multiplication strategies to explain and illustrate a calculation. For example, when solving 43×22, draw an area model of $(40 + 3) \times (20 + 2)$. Connect this area model to the partial product expression $(40 \times 20) + (3 \times 20) + (40 \times 2) + (3 \times 2)$. Additionally, simplify the partial product expression in order to find the value of the original expression. Further, given an area model, reconstruct the original multiplication expression.

- Ask students to construct an area model for a given multiplication problem by decomposing the factors based on place value. For example, given the problem 28×35, decompose the factors into 20 and 8 (for 28) and 30 and 5 (for 35). The area model shown represents the product, $(20 + 8) \times (30 + 5)$, using these decomposed factors. The partial product expression is $(20 \times 30) + (8 \times 30) + (20 \times 5) + (8 \times 5)$.



28 × 35 = 600 + 240 + 100 + 40 = 980

- Ask students to write the original expression from a given area model as well as the partial product expression and then use the partial product expression to determine the product. For example, consider the area model shown.



The original expression is 35×62, since $30 + 5 = 35$ and $60 + 2 = 62$. The partial product expression is $(30 \times 60) + (30 \times 2) + (5 \times 60) + (5 \times 2)$. Evaluating each part of the partial product expression gives the sum $1800 + 60 + 300 + 10$, which is equal to 2,170. Therefore, the product is $35 \times 62 = 2{,}170$.

How do knowledge of place value and properties of operations help solve multiplication problems?

M.P.2. Reason abstractly and quantitatively. Decompose and compose numbers in a variety of ways using place value and the properties of operations to demonstrate a variety of different strategies that use known facts. For example, multiplication of a multi-digit number by a one-digit number can be simplified into a one-digit number by one-digit number multiplication problem multiplied by a multiple of 10, such as the multiplication problem $364 \times 8 = (3 \times 8 \times 100) + (6 \times 8 \times 10) + (4 \times 8)$. Additionally, use the distributive property to rewrite multiplication problems. For example, given 24×35, rewrite the product as $(24 \times 30) + (24 \times 5)$ or $(35 \times 20) + (35 \times 4)$.

- Ask students to rewrite the product of a three-digit number and a single-digit number using properties of operations and place value. For example, 581×7 is equivalent to $(500 \times 7) + (80 \times 7) + (1 \times 7)$. The first 2 products in this expression can be decomposed further using place value to create the equivalent expression $(5 \times 100 \times 7) + (8 \times 10 \times 7) + (1 \times 7)$. Each of these products can be evaluated using single-digit multiplication combined with an understanding of place value to get $3{,}500 + 560 + 7$. Therefore, $581 \times 7 = 4{,}067$.

- Ask students to rewrite the product of 2 two-digit numbers using properties of operations. For example, 73×56 is equivalent to $(70 + 3) \times 56$, which can be rewritten as $(70 \times 56) + (3 \times 56)$. Each of these products can be decomposed further to create, for example, the expression $(70 \times 50) + (70 \times 6) + (3 \times 50) + (3 \times 6)$. Simplifying these products gives $3{,}500 + 420 + 150 + 18$. Therefore, $73 \times 56 = 4{,}088$.

What strategy makes the most sense to use to solve a given multiplication problem and why?

M.P.3. Construct viable arguments and critique the reasoning of others. Analyze a given problem and choose the strategy that provides an entry point. Depending on experiences and level of proficiency, one problem could have many different strategies that would all be appropriate. For example, when solving 35×14, one way is to decompose 35 into $30 + 5$ and 14 into $10 + 4$ and solve the problem using four different products. Another way to solve the same problem is by doubling/halving to adjust the problem from 35×14 to 70×7. Additionally, explain why the chosen strategy works for a specific problem and observe that there are other possible valid strategies.

- Ask students to rewrite a product in which one of the factors is close to a number that ends in zero and then use properties of operations to find the product. For example, 49×82 is equivalent to $(50 \times 82) – 82$ since $49 = 50 – 1$. The product 50×82 is equivalent to $50 \times 80 + 50 \times 2$, which gives 4,100. Therefore, $49 \times 82 = 4{,}100 – 82$ which is equal to 4,018.

- Ask students to describe valid strategies for multiplication problems in which one of the factors is close to a round number. For example, given 58×34, students might describe the strategy of rounding 58 up to 60, performing the multiplication, and then subtracting 68 from the product, since $2 \times 34 = 68$. Likewise, given 91×76, students might describe the strategy of rounding 91 down to 90, performing the multiplication, and then adding 76 to the product, since $1 \times 76 = 76$.

- Ask students to describe valid strategies using decomposition of one or both of the factors. For example, when solving 42×73, students might describe the strategy of decomposing one or both of the factors.
    - Decomposing 42 into $40 + 2$ allows the product to be rewritten as $(40 \times 73) + (2 \times 73)$.
    - Decomposing 73 into $70 + 3$ allows the product to be rewritten as $(42 \times 70) + (42 \times 3)$.
    - Decomposing both 42 and 73 allows the product to be written as $(40 \times 70) + (40 \times 3) + (2 \times 70) + (2 \times 3)$.
- Ask students to describe valid strategies using doubling or halving of one of the factors. For example, given 45×38, students might find the product 90×38 and then take half of the product, since half of 90 is 45. Likewise, given 82×56, students might find the product 41×56 and then double the product, since 41×2 is 82. In order to find either of these products, students might use other strategies such as rounding one of the factors or decomposing one or both of the factors.

How is multiplying a two-, three-, or four-digit number by a one-digit number similar to and different from multiplying a two-digit number by a two-digit number?

M.P.8. Look for and express regularity in repeated reasoning. Connect multiplying by a one-digit number to multiplying by a two-digit number to eventually lead to a general method using knowledge of place value and the properties of operations instead of using the standard algorithm for multiplication, which is introduced in grade 5. For example, compare multiplying 42×8 to multiplying 42×18 and note similarities and differences in the process. Additionally, explain how to use smaller products to determine the value of products that involve multiples of 10.

- Ask students to calculate two related multiplication problems and note similarities and differences in the products. For example, ask students to calculate the products 56×9 and 56×29. Students should note that one step in calculating 56×29 is to calculate 56×9. Likewise, given 56×9 and 56×93, students should note that one step in calculating 56×93 is to multiply 56×90. The product 56×90 can be rewritten as $(56 \times 9) \times 10$.

- Ask students to use a given product to determine the value of the product when one factor has been multiplied by ten. For example, ask students to use the product 86×5 to determine the value of 86×50. Because $86 \times 5 = 430$ and the product 86×50 can be expressed as $(86 \times 5) \times 10$, the value of $86 \times 50 = 430 \times 10$. Students should identify this as 430 tens, which has a value of 4,300. Therefore, $86 \times 50 = 4{,}300$.

Key Academic Terms:

strategy, multiply, place value, equation, array, area model, partial product, multiple of 10, factor, expression, double, half, repeated reasoning, similarities, differences, properties of operations, product

Additional Resources:

- Article: Exploring two-digit multiplication with base-ten blocks
- Games: Multiplication games
- Lesson: Modeling multiplication
- Article: Arrays and area models to the standard algorithm
- Article: Strategies for teaching multi-digit multiplication

12a

Operations with Numbers: Base Ten
Use place value understanding and properties of operations to perform multi-digit arithmetic with whole numbers.
12. Use strategies based on place value, properties of operations, and/or the relationship between multiplication and division to find whole-number quotients and remainders with one-digit divisors and up to four-digit dividends. a. Illustrate and/or explain quotients using equations, rectangular arrays, and/or area models.

Guiding Questions with Connections to Mathematical Practices:

How can multiplication be used to solve division problems?

M.P.2. Reason abstractly and quantitatively. Describe division as an unknown factor problem, and use the greatest multiple strategy to solve. For example, when solving $122 \div 3$, rewrite as $3 \times n = 122$. The greatest number of groups of size 3 that are in 12 tens and 2 ones is 4 tens with 2 ones remaining, so the answer is 40 with a remainder of 2. Additionally, identify that the order of the divisor and dividend in a division problem determine which value is a factor and which value is the product in the related multiplication problem.

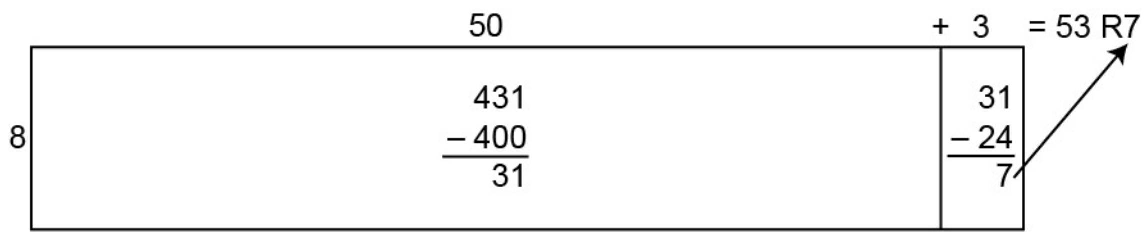
- Ask students to rewrite division problems as related multiplication problems, using a variable to represent the unknown quotient. For example, the quotient $243 \div 5$ can be reworded as “How many groups of 5 are in 243?” which is equivalent to asking, “What number times 5 is equal to 243?” Translate the statement into a multiplication problem, using a variable to represent the unknown number. In this case, the multiplication problem is $n \times 5 = 243$ or $5 \times n = 243$.

- Ask students to use their knowledge of multiples to solve division problems when rewritten as related multiplication problems. For example, the quotient $207 \div 4$ can be rewritten as $4 \times n = 207$. To solve this multiplication equation, determine the greatest number of groups of size 4 that are in 20 tens and 7 ones. There are 5 groups of size 4 in 20 and therefore 50 groups of size 4 in 20 tens. There is 1 group of size 4 in 7 with 3 remaining, so the answer is 51 with a remainder of 3.

How are area models related to equations and division strategies?

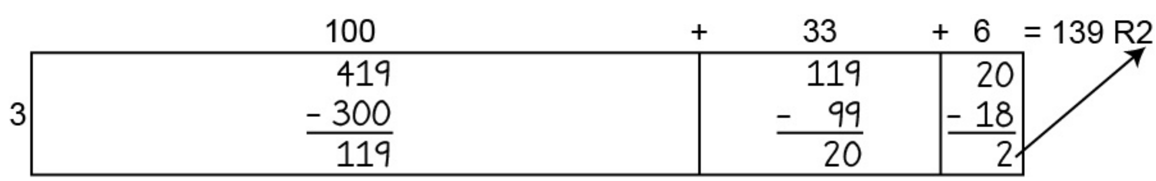
M.P.4. Model with mathematics. Connect area models to equations and division strategies to explain and illustrate a calculation. For example, when solving $762 \div 6$, use an area model to show that for a rectangle with a total area of 762 and a side length of 6, the length of the unknown side is 1 hundred, 2 tens, and 7 ones. Show this by decomposing 762 to $600 + 120 + 42$. Then write equations that correspond with the model (e.g., $762 - (6 \times 100) = 162$; $162 - (6 \times 20) = 42$; $42 - (6 \times 7) = 0$). Additionally, show that there are multiple different area models for a given division problem but each area model gives the same quotient.

- Ask students to create and use an area model to find a given quotient. For example, the following area model shows one method of determining the quotient $431 \div 8$.



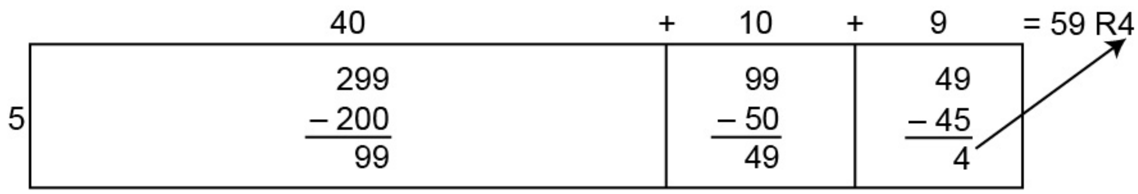
Because there are at least 50 groups of size 8 in 431, subtract the product $8 \times 50 = 400$ from 431, leaving 31. Then, because there are at least 3 groups of size 8 in 31, subtract the product $8 \times 3 = 24$ from 31, leaving 7 remaining. Since 8 is greater than 7, the remainder is 7, while the rest of the quotient is equal to the sum of 50 and 3.

- Ask students to write an equation that corresponds to each box of a division model. For example, the division model shown represents the quotient $419 \div 3$.

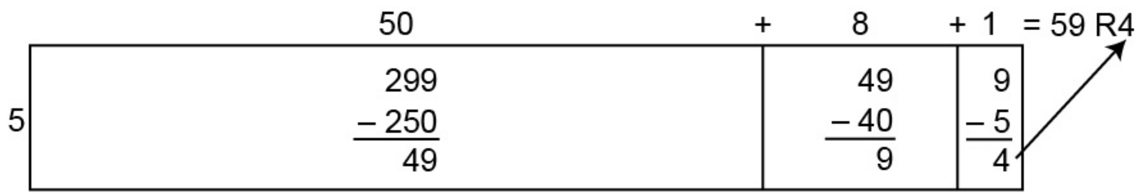


To create the equations that correspond with the model, start with the minuend, 419, in the first box. To determine how to write the subtrahend of 300, recall that the subtrahend was determined by multiplying 3×100. Therefore, the equation $419 - (3 \times 100) = 119$ corresponds to the first box in the model. Similarly, the equation $119 - (3 \times 33) = 20$ corresponds to the second box in the model. Finally, the equation $20 - (3 \times 6) = 2$ corresponds to the last box in the model. The value of the difference for the last box in the model should equal the remainder; in this case, the remainder is 2.

- Ask students to create multiple different models for a given division problem and show that each area model gives the same quotient. For example, the first area model shows one method of determining the quotient 299 ÷ 5.



The second area model shows a different method to determine the same quotient.



Students should notice that the models have different quantities being subtracted from the dividend and different numbers being composed to make the quotient, but the sum of the numbers being composed (and the remainder) result in the same quotient.

How do knowledge of place value and properties of operations help solve division problems?

M.P.2. Reason abstractly and quantitatively. Decompose and compose numbers in a variety of ways using place value and the properties of operations to demonstrate a variety of different strategies for division. For example, use place value knowledge when solving 3,600 ÷ 9 to show that 3,600 is the same as 36 hundreds and that 36 hundreds ÷ 9 is 4 hundreds, or 400. Additionally, observe that decomposing in different ways may result in different division computations but that the quotient remains the same, regardless of the way it is decomposed or calculated.

- Ask students to use place value to divide multi-digit whole numbers ending in one or multiple zeros by a single-digit number. For example, in the quotient 2,800 ÷ 4, the number 2,800 can be thought of as 28 hundreds. Because $28 \div 4 = 7$, 28 hundreds divided by 4 is equal to 7 hundreds. Therefore, $2{,}800 \div 4 = 700$. Likewise, in the quotient 9,000 ÷ 3, the number 9,000 can be thought of as 9 thousands. Because $9 \div 3 = 3$, 9 thousands divided by 3 is equal to 3 thousands. Therefore, $9{,}000 \div 3 = 3{,}000$.

- Ask students to decompose dividends to break a larger division problem down into smaller division problems. For example, in the quotient $2{,}466 \div 6$, the number 2,466 can be decomposed into $2{,}400 + 66$. Then, each of these addends can be divided by 6.

$$2{,}466 \div 6$$

$$2{,}400 \div 6 = 400$$

$$66 \div 6 = 11$$

Therefore, the quotient $2{,}466 \div 6 = 400 + 11 = 411$. In this quotient, students could also decompose the 66 further, into $60 + 6$, to show that $2{,}466 \div 6 = 400 + 10 + 1 = 411$.

How does dividing a two-, three-, or four-digit number by a one-digit number connect to dividing numbers within 100?

M.P.8. Look for and express regularity in repeated reasoning. Connect knowledge of dividing numbers within 100 to dividing larger numbers using knowledge of place value, the properties of operations, and the relationship between multiplication and division. This prepares students to learn the standard algorithm for division, which is introduced in grade 6. For example, compute $87 \div 4$, $320 \div 3$, and $2{,}444 \div 6$, noting similarities and differences in the division. Additionally, identify how decomposing a multi-digit number can allow for the use of division facts within 100.

- Ask students to find two quotients in which the two dividends share common leading digits. For example, ask students to find the quotients $92 \div 4$ and $928 \div 4$. Once students have determined that $92 \div 4 = 23$ (possibly by decomposing 92 into $80 + 12$, for example), decompose 928 into $920 + 8$. Use the fact that $92 \div 4 = 23$, along with the fact that $8 \div 4 = 2$, to show that $928 \div 4 = 232$.

- Ask students to use decomposition to rewrite a dividend in such a way as to allow the use of division facts within 100. For example, given $2{,}884 \div 7$, the dividend can be rewritten as $2{,}800 + 70 + 14$. Since each addend must be divided by 7, the number facts $28 \div 7 = 4$, $7 \div 7 = 1$, and $14 \div 7 = 2$ can all be used, along with place value, to calculate the original quotient.

$$2{,}884 \div 7$$

  - The quotient $2{,}800 \div 7 = 400$ because 28 hundreds $\div\ 7 = 4$ hundreds.
  - The quotient $70 \div 7 = 10$ because 7 tens $\div\ 7 = 1$ ten.
  - The quotient $14 \div 7 = 2$.

Therefore, the quotient $2{,}884 \div 7 = 412$.

Key Academic Terms:

quotient, dividend, divisor, divide, multiply, multiple, equation, remainder, area model, greatest multiple, decompose, compose, array, properties of operations, unknown factor, variable

Additional Resources:

- Article: Learning long division
- Activity: Mental division strategy

13a

Operations with Numbers: Fractions
Extend understanding of fraction equivalence and ordering. *Denominators are limited to 2, 3, 4, 5, 6, 8, 10, 12, and 100.*
13. Using area and length fraction models, explain why one fraction is equivalent to another, taking into account that the number and size of the parts differ even though the two fractions themselves are the same size. a. Apply principles of fraction equivalence to recognize and generate equivalent fractions. *Example*: $\frac{a}{b}$ is equivalent to $\frac{n \times a}{n \times b}$.

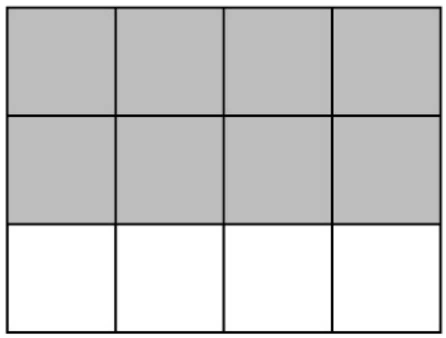
Guiding Questions with Connections to Mathematical Practices:

How can fractions with different denominators and numerators be equivalent?

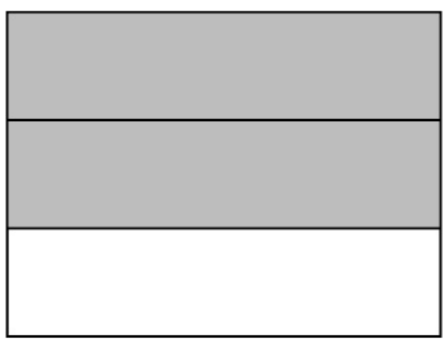
M.P.3. Construct viable arguments and critique the reasoning of others. Demonstrate that two fractions are equivalent, using various fraction models, and demonstrate that even though the number and size of the parts differ, the fractions are the same size. For example, use an area model to explain that $\frac{6}{8} = \frac{3}{4}$ because both fractions shade the same amount of area on the model. Additionally, use two number lines to show that $\frac{1}{5} = \frac{2}{10}$ because both are at the same location on the number line.

- Ask students to use fraction circles and paper strips to create and find equivalence with two different denominators. Students may be given one fraction, such as $\frac{3}{5}$, and asked to find any equivalent fractions that have denominators of 2, 3, 4, 6, 8, 10, 12, and 100 by placing fraction circle pieces of those denominators on top of the $\frac{3}{5}$ fraction circle pieces to see if they have equal areas. Students then write the fraction equivalence in equations, such as $\frac{3}{5} = \frac{6}{10} = \frac{60}{100}$ to connect the concrete ideas to the abstract ones. Students may make informal conclusions with complex fractions, like $\frac{1}{2} = \frac{2.5}{5}$, which should be acknowledged but not formally taught.

- Ask students to use drawn models such as rectangles, circles, tape diagrams, and number lines to find equivalent fractions. Students should experience finding equivalent fractions with denominators greater than the given fraction by adding partitions and finding equivalent fractions with denominators less than the given fraction by removing or imagining removing partitions. For example, given the fraction $\frac{8}{12}$, students draw the rectangle shown.



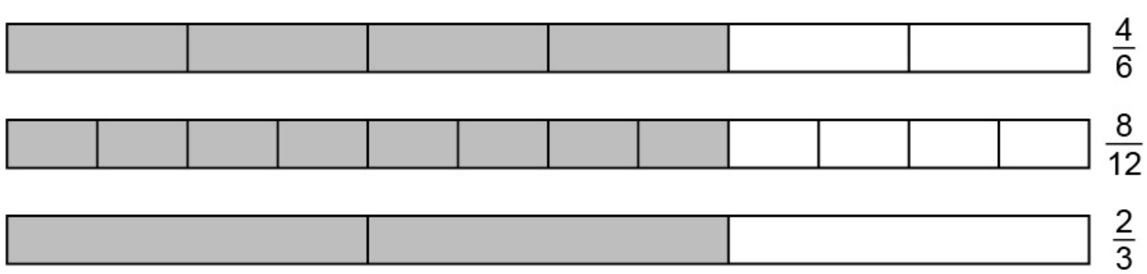
The students then redraw the rectangle removing some of the partitions, as shown, to find the equivalent fraction $\frac{2}{3}$.



How can equivalence be determined for two fractions?

M.P.8. Look for and express regularity in repeated reasoning. Using visual fraction models, determine that, when a fraction's numerator and denominator are multiplied or divided by $\frac{n}{n}$ to create a new fraction, the new fraction is equivalent to the original. For example, use a tape diagram to find that $\frac{6}{8}$ is equivalent to both $\frac{3}{4}$ and $\frac{12}{16}$ because $\frac{6 \div 2}{8 \div 2} = \frac{3}{4}$ and $\frac{6 \times 2}{8 \times 2} = \frac{12}{16}$, therefore, $\frac{3}{4} = \frac{6}{8} = \frac{12}{16}$. Additionally, use fraction circles to find multiple fractions equivalent to $\frac{2}{3}$ by placing two $\frac{1}{3}$ fraction circles and finding that eight $\frac{1}{12}$ fraction circles $(\frac{8}{12})$ are equivalent, and explain that because four $\frac{1}{12}$ pieces fit on each $\frac{1}{3}$ piece, the numerator and denominator of $\frac{2}{3}$ are each multiplied by 4 to get $\frac{8}{12}$.

- Ask students to use a variety of hands-on manipulatives, drawn models, and equations to explore fraction equivalence. For example, students explore the fraction $\frac{4}{6}$ by creating paper strips to find equivalent fractions with both greater and lesser denominators. Write equations to represent the paper strips and the fraction equivalent to 1 that they multiplied or divided by to get the equivalent fraction. Explain that $\frac{4}{6}$ can be folded so that each sixth is doubled to be a denominator of twelve, therefore the 4 sixths were also doubled for a numerator of eight.



$$\frac{4 \times 2}{6 \times 2} = \frac{8}{12}$$

$$\frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

- Ask students to understand that any fraction with the same numerator and denominator is equivalent to 1. Therefore, when a fraction is multiplied by another fraction equivalent to 1, the value of the number doesn't change, just like with whole numbers. Also, multiply whole numbers by a fraction equivalent to one, such as $5 \times \frac{12}{12}$ is equal to 5 or $\frac{60}{12}$, to help generalize the meaning of a fraction equivalent to 1.

- Ask students to determine equivalence by determining whether the fraction multiplier is equivalent to 1. For example, ask students "Are the fractions $\frac{3}{5}$ and $\frac{6}{8}$ equivalent? Use a fraction equivalent to 1 to explain your solutions." Students should explain that the fractions are not equivalent, because even though the differences between the numerators and denominators are both 3, when $\frac{3}{5}$ is multiplied by $\frac{2}{2}$, the denominator would be 10, not 8, which it would be if the fractions were equivalent.

Key Academic Terms:

fraction, numerator, denominator, equivalent, fraction models, multiples, multiply, area model, tape diagram, area fraction model

Additional Resources:

- Book: Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: fractions & decimals*. Portsmouth, NH: Heinemann.

- Activity: BINGO fractions math game

14a

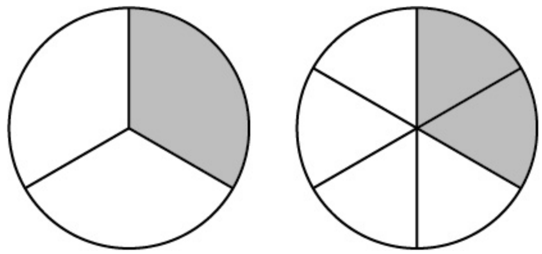
Operations with Numbers: Fractions
Extend understanding of fraction equivalence and ordering. *Denominators are limited to 2, 3, 4, 5, 6, 8, 10, 12, and 100.*
14. Compare two fractions with different numerators and different denominators using concrete models, benchmarks (0, $\frac{1}{2}$, 1), common denominators, and/or common numerators, recording the comparisons with symbols >, =, or <, and justifying the conclusions. a. Explain that comparison of two fractions is valid only when the two fractions refer to the same whole.

Guiding Questions with Connections to Mathematical Practices:

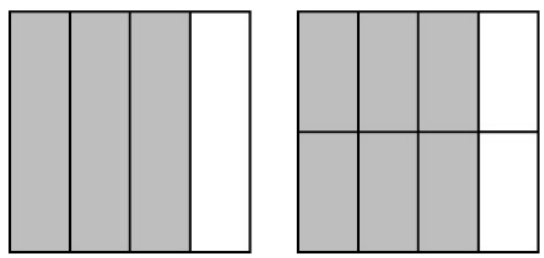
How can fractions with unlike numerators and denominators be compared?

M.P.1. Make sense of problems and persevere in solving them. Choose a strategy that makes the most sense, such as finding a common numerator or denominator, using a benchmark fraction, or using residual thinking, to compare two given fractions. For example, when comparing $\frac{13}{12}$ and $\frac{6}{5}$, students should notice that both are one fractional part larger than one whole ($\frac{12}{12} + \frac{1}{12}$ and $\frac{5}{5} + \frac{1}{5}$) and $\frac{1}{5}$ is larger than $\frac{1}{12}$, so $\frac{6}{5}$ is greater than $\frac{13}{12}$. Additionally, use geometric models and number lines to compare fractions.

- Ask students to use geometric models and number lines to recognize and identify equivalent fractions when one denominator is a factor of the other denominator. Start with a geometric model of the fraction with the smaller denominator, and then divide each of the parts to model a fraction that is equivalent to the fraction with the larger denominator. For example, models for $\frac{1}{3} = \frac{2}{6}$ and $\frac{3}{4} = \frac{6}{8}$ are shown.

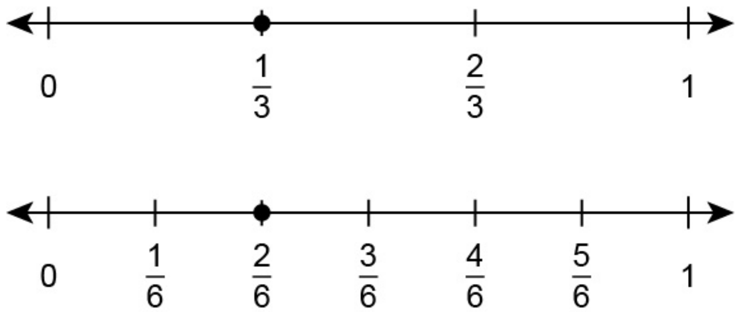


$\frac{1}{3} = \frac{2}{6}$

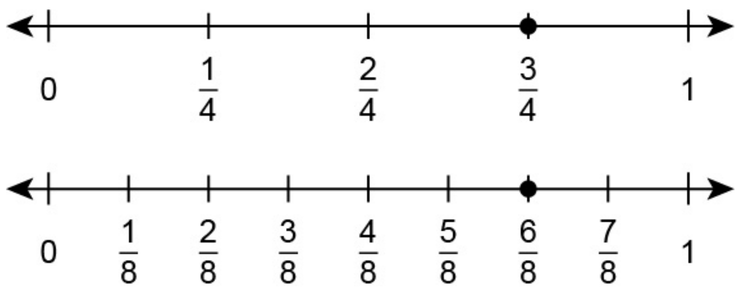


$\frac{3}{4} = \frac{6}{8}$

Demonstrate that when two number lines are vertically aligned, equivalent fractions with different denominators are also vertically aligned. The number lines shown also depict that $\frac{1}{3} = \frac{2}{6}$ and that $\frac{3}{4} = \frac{6}{8}$.



$\frac{1}{3} = \frac{2}{6}$



$\frac{3}{4} = \frac{6}{8}$

- To compare fractions with unlike denominators, ask students to find common denominators by creating lists of multiples of the denominators. For example, the fractions $\frac{3}{4}$ and $\frac{1}{3}$ have denominators of 4 and 3.

The multiples of 4 are 4, 8, 12, 16, 20 . . .

The multiples of 3 are 3, 6, 9, 12, 15 . . .

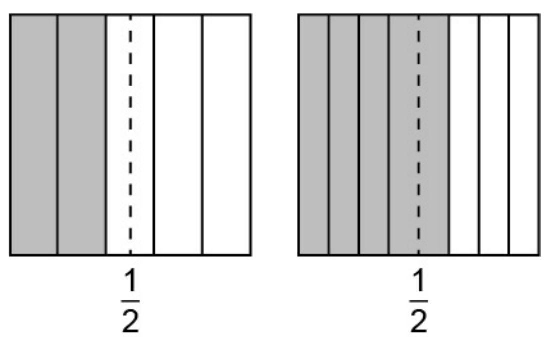
A common multiple is 12, so multiply by a fraction equivalent to 1 to find equivalent fractions with a denominator of 12.

$$\frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$$

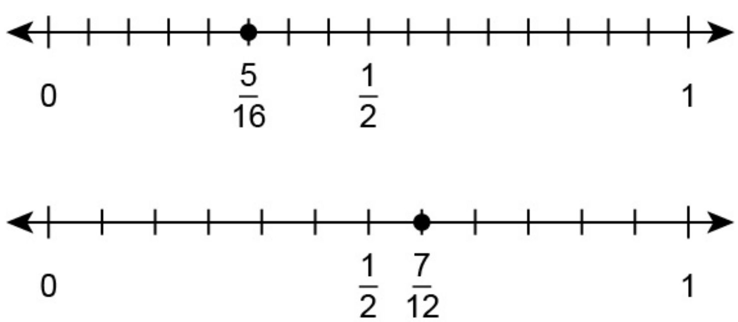
$$\frac{1}{3} \times \frac{4}{4} = \frac{4}{12}$$

Since $\frac{9}{12}$ is greater than $\frac{4}{12}$, $\frac{3}{4}$ is greater than $\frac{1}{3}$.

- To compare fractions by first comparing each fraction to $\frac{1}{2}$, ask students to create a rectangular model for a fraction that is less than $\frac{1}{2}$ and for a fraction that is greater than $\frac{1}{2}$. For both models, represent $\frac{1}{2}$ with a dashed line. Then compare the fractions with $\frac{1}{2}$ and use the results to compare the fractions with each other. For example, the geometric models (squares) show that $\frac{2}{5} < \frac{1}{2}$ and that $\frac{5}{8} > \frac{1}{2}$. Therefore, $\frac{2}{5} < \frac{5}{8}$.



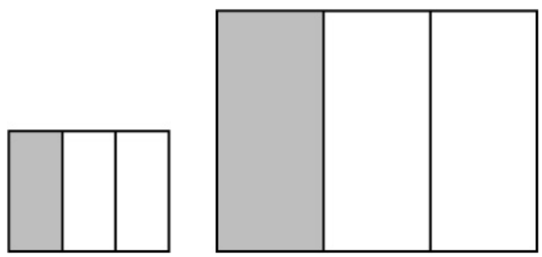
The same comparisons can be made with a number line. For example, the number lines show that $\frac{5}{16} < \frac{1}{2}$ and that $\frac{7}{12} > \frac{1}{2}$. Therefore, $\frac{5}{16} < \frac{7}{12}$.



When is it appropriate/not appropriate to compare two fractions?

M.P.2. Reason abstractly and quantitatively. Determine whether two fractions are describing the same whole before comparing them. For example, $\frac{3}{4}$ of a small pizza is not the same as $\frac{9}{12}$ of a large pizza even though $\frac{3}{4} = \frac{9}{12}$ because the whole isn't the same. Additionally, $\frac{1}{2}$ of an apple is not the same as $\frac{1}{2}$ of a grape.

- Ask students to use two different fraction models with obviously different-sized wholes to illustrate that two fractions with the same numerators and denominators are equivalent only when referring to the same whole. In the example shown, the two fractions cannot be compared, because they refer to different wholes.

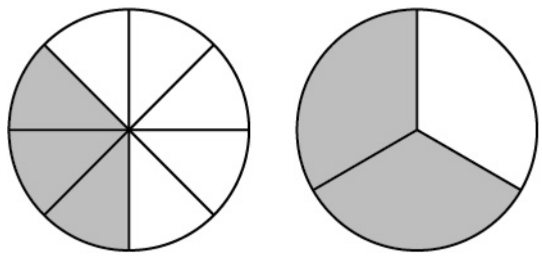


- Ask students to discuss real-world comparisons between fractions. It may be appropriate to compare the fraction remaining of two different pizzas if the two pizzas are the same size, but it will not be appropriate when the pizzas are different sizes. Comparing a fraction of an orange to a fraction of a watermelon does not make sense unless a common whole can be created (e.g., $\frac{1}{2}$ of an orange cannot be compared to $\frac{1}{8}$ of a watermelon, but by introducing weight in pounds as a common whole, a valid comparison, such as $\frac{3}{16}$ pound of orange is less than $\frac{15}{16}$ pound of watermelon, can be made).

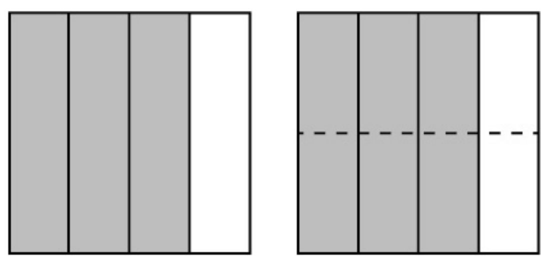
How can symbols be used to record the comparison of two fractions with unlike numerators and denominators?

M.P.6. Attend to precision. Record the comparison of two fractions using $<$, $>$, or $=$. For example, $\frac{5}{6} < \frac{11}{12}$. Additionally, $\frac{1}{2} > \frac{2}{5}$.

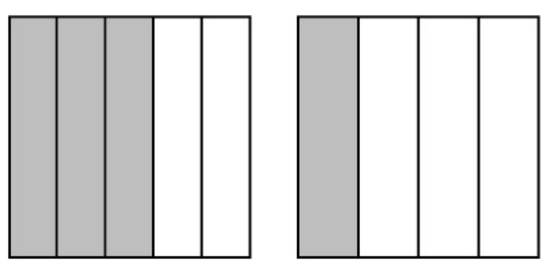
- Ask students to record the result of a comparison by using the symbols $<$, $=$, and $>$. For the examples shown, record the results as $\frac{3}{8} < \frac{2}{3}$, $\frac{3}{4} = \frac{6}{8}$, and $\frac{3}{5} > \frac{1}{4}$.



$\frac{3}{8} < \frac{2}{3}$

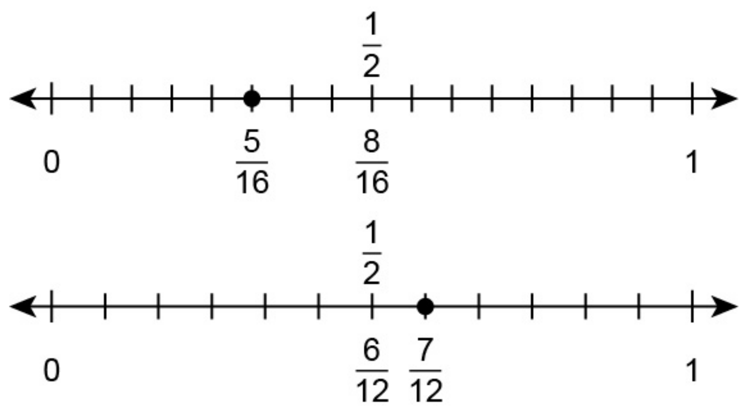


$\frac{3}{4} = \frac{6}{8}$

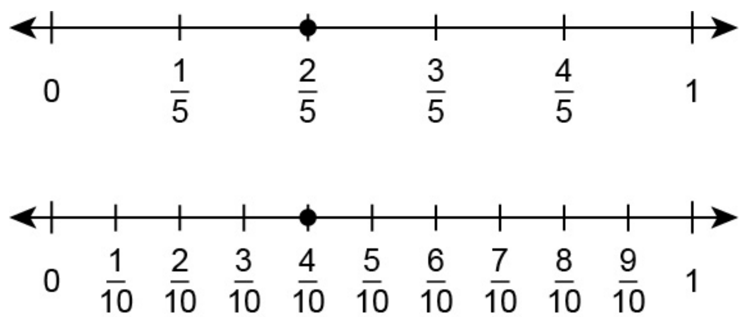


$\frac{3}{5} > \frac{1}{4}$

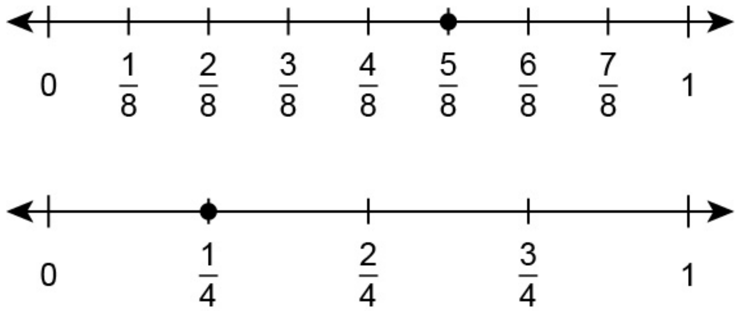
- Ask students to show that less than ($<$) means “to the left of” on a number line, equals ($=$) means “in the same location as” on a number line, and greater than ($>$) means “to the right of” on a number line by asking students to represent these relationships on pairs of vertically aligned number lines. For example, the comparisons $\frac{5}{16} < \frac{7}{12}$, $\frac{2}{5} = \frac{4}{10}$, and $\frac{5}{8} > \frac{1}{4}$ are represented on number lines as shown.



$\frac{5}{16} < \frac{7}{12}$



$\frac{2}{5} = \frac{4}{10}$

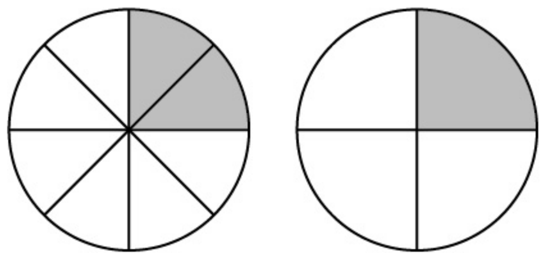


$\frac{5}{8} > \frac{1}{4}$

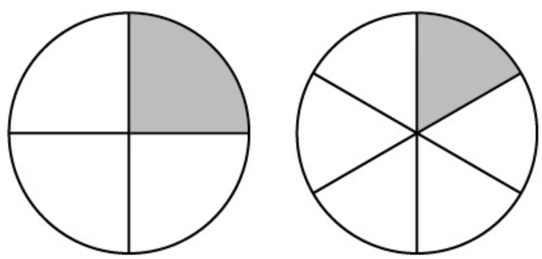
In what ways can the comparison of two fractions be justified?

M.P.3. Construct viable arguments and critique the reasoning of others. Justify comparisons using objects, drawings, diagrams, equations, and/or words. For example, when comparing $\frac{2}{6}$ and $\frac{1}{5}$, students may use fraction circles, tape diagrams, or words to find $\frac{2}{6} = \frac{1}{3}$, then use the same-numerator strategy to conclude $\frac{1}{3} > \frac{1}{5}$; therefore, $\frac{2}{6} > \frac{1}{5}$. Additionally, when comparing $\frac{4}{5}$ and $\frac{5}{6}$, a student may use fraction strips to determine that out of a same-sized whole, the unshaded part of a fraction strip representing $\frac{4}{5}$ is greater than the unshaded part of a fraction strip representing $\frac{5}{6}$; therefore, $\frac{5}{6} > \frac{4}{5}$.

- Ask students to explain or demonstrate their reasoning behind the comparison of two fractions. For example, to show $\frac{2}{8} > \frac{1}{6}$, first show that $\frac{2}{8} = \frac{1}{4}$ using circle fraction models as shown. Then, explain using words that $\frac{1}{4} > \frac{1}{6}$ because when a whole is divided into 4 equal-sized pieces, each of those pieces is larger than the pieces when the whole is divided into 6 equal-sized pieces. Therefore, $\frac{2}{8} > \frac{1}{6}$.



$\frac{2}{8} = \frac{1}{4}$



$\frac{1}{4} > \frac{1}{6}$

- Ask students to justify reasoning using manipulatives like fraction strips to compare two fractions. For example, to compare $\frac{7}{8}$ and $\frac{9}{10}$, first show that the remaining part of the whole strip for $\frac{7}{8}$ is $\frac{1}{8}$ and the remaining part of $\frac{9}{10}$ is $\frac{1}{10}$. Since the remaining part $\frac{1}{8}$ is larger than the remaining part $\frac{1}{10}$, it follows that the fraction $\frac{7}{8} < \frac{9}{10}$.

$$\frac{7}{8} < \frac{9}{10}$$

Key Academic Terms:

compare, equivalent fractions, numerator, denominator, benchmark fraction, less than (<), greater than (>), equal to (=), common numerator, common denominator, tape diagram, equation, residual thinking, fraction circle, model, whole, number line, multiples, vertically, common multiple

Additional Resources:

- Book: Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: fractions & decimals*. Portsmouth, NH: Heinemann.
- Article: [Ways to compare fractions]

15a

Operations with Numbers: Fractions
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
15. Model and justify decompositions of fractions and explain addition and subtraction of fractions as joining or separating parts referring to the same whole. a. Decompose a fraction as a sum of unit fractions and as a sum of fractions with the same denominator in more than one way using area models, length models, and equations.

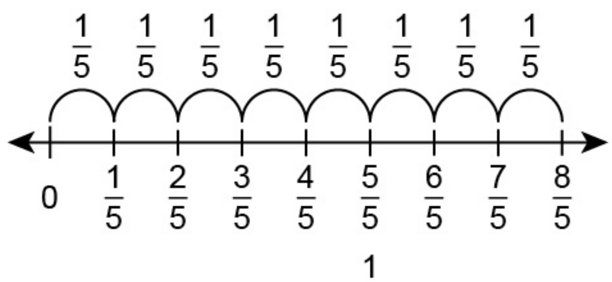
Guiding Questions with Connections to Mathematical Practices:

How many of $\frac{1}{b}$ are added to compose $\frac{a}{b}$?

M.P.2. Reason abstractly and quantitatively. Explain that any fraction, $\frac{a}{b}$, is the sum of *a* of the unit fraction $\frac{1}{b}$. For example, $\frac{3}{2}$ is the sum of 3 of the unit fraction $\frac{1}{2}$. Additionally, the sum can be written as $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$ or $\frac{1+1+1}{2}$.

- Ask students to name the unit fraction within any fraction and repeatedly add it to decompose the fraction. Given the fraction $\frac{5}{12}$, name the unit fraction as $\frac{1}{12}$ and then repeatedly add $\frac{1}{12}$ a total of five times to decompose the fraction.

- Ask students to use models such as tape diagrams, fraction circles, number lines, and rectangles to represent a fraction as a sum of unit fractions. In the diagram shown, represent the fraction $\frac{8}{5}$ using a number line by making 8 jumps of $\frac{1}{5}$.



Then, represent the model using expressions and equations. For this example, write the following:

$$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{8}{5}$$

How is adding and subtracting fractions with the same denominator connected to adding and subtracting whole numbers?

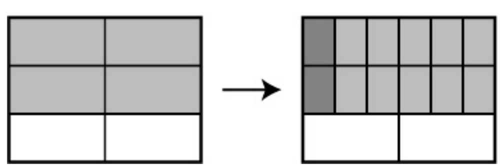
M.P.7. Look for and make use of structure. Explain that addition is joining parts and subtraction is separating parts for both whole numbers and fractions. For example, representing $2+3$ on a number line as joining 2 units and 3 units is similar to representing $\frac{2}{8}+\frac{3}{8}$ as joining $\frac{2}{8}$ of a unit and $\frac{3}{8}$ of the same unit. Additionally, use a tape diagram to model that subtracting 1 hundred from 3 hundreds (with the hundred being the unit) results in 2 hundreds, or 200. Further, use a tape diagram to model that subtracting 1 one-sixth from 3 one-sixths (with $\frac{1}{6}$ being the unit) results in 2 one-sixths, or $\frac{2}{6}$.

- Ask students to determine whether addition or subtraction should be used to solve a given word problem. For example, Estelle's dog requires $\frac{3}{10}$ pound of food per day and her cat requires $\frac{1}{10}$ pound of food per day. How much food does Estelle need each day for both her pets? Discuss with students what operation should be used and how they know. If students have trouble recognizing that addition is required, give them a simpler scenario: Estelle's dog requires 2 pounds of food per day and her cat requires 1 pound of food per day. Help students to recognize that the same patterns that indicate addition or subtraction with whole numbers still apply with fractions.
- Ask students to interpret the meaning of a sum or difference in a word problem. For example, Dorian has $\frac{7}{4}$ liter of paint. After painting his bathroom, he has $\frac{1}{4}$ liter of paint left. What does the difference $\frac{7}{4} - \frac{1}{4}$ represent in this context? Discuss why it represents the amount of paint that Dorian used and draw parallels to similar problems involving whole numbers.
- Ask students to generalize commonalities between whole number operations and fraction operations. After multiple experiences adding and subtracting different unit fractions and discussions about place value, state that addition is always joining parts together and subtraction is always separating parts. Students will also note that in addition, numbers can always be added in any order, but this is not true in subtraction. It is important to note that misconceptions, such as "addition makes larger" and "always subtract the smaller number from the bigger number" should not be reinforced.

When is it appropriate/not appropriate to add and subtract fractions?

M.P.3. Construct viable arguments and critique the reasoning of others. Create examples and non-examples of situations in which adding and subtracting fractions is appropriate. For example, when a student is given fractions without context, the unit is assumed to be the same whole for all of them, and those fractions can always be added and subtracted no matter the denominator. A counterexample is the following situation: "Jayquan has $\frac{3}{4}$ of a cup of milk. He drinks $\frac{1}{4}$ of his milk. How much milk remains?" This problem cannot be solved by subtracting the fractions because $\frac{1}{4}$ of $\frac{3}{4}$ cup of milk is not equal to $\frac{1}{4}$ cup. Additionally, when given the expression $\frac{9}{10} - \frac{7}{10}$, it is assumed that the wholes are the same, so the difference between $\frac{9}{10}$ and $\frac{7}{10}$ is $\frac{2}{10}$.

- Ask students to determine whether it is appropriate to add or subtract the fractions in word problems and problems without context. Students should determine that problems without any context always refer to the same whole and can be added or subtracted as they are. Students may analyze word problems by drawing pictures to help determine whether the word problem is referring to the same whole. For example, the story problem "There is $\frac{4}{6}$ of a pan of brownies. Marcus gives $\frac{1}{6}$ of the brownies in the pan to his friend." can be solved if students recognize that the fraction $\frac{4}{6}$ refers to the space of the entire pan while the fraction $\frac{1}{6}$ refers to the quantity of brownies. The students may also draw the picture shown to illustrate that $\frac{1}{6}$ size pieces of the whole are not being given away, but $\frac{1}{6}$ of the $\frac{4}{6}$ of the brownies are being given away.



Note: The students are not required to solve the problem.

- Ask students to explain why it is not appropriate to add or subtract fractions in certain scenarios. For example, if Boris has $\frac{3}{4}$ foot of ribbon and $\frac{1}{4}$ yard of satin, how much decorating material does he have in total? The fractions cannot be added because in one case the whole unit is a foot and in the other the whole unit is a yard.

- Ask students to write simple addition and subtraction story problems that use the same whole. For example, given the expression $\frac{7}{8} - \frac{2}{8}$ students may correctly write, "There is $\frac{7}{8}$ of a blueberry pie. Xia eats $\frac{2}{8}$ of the whole pie." An incorrect story problem example would be "A bus of students is $\frac{7}{8}$ full and $\frac{2}{8}$ of those students get off the bus" as the $\frac{2}{8}$ of those students is not referring to the whole bus, but the $\frac{7}{8}$ of the students on the bus.

How is decomposing a fraction connected to decomposing a whole number?

M.P.7. Look for and make use of structure. Explain the connection between the decomposition of fractions and whole numbers. For example, both whole numbers and fractions can be decomposed into addition equations in a variety of ways. Additionally, create and test generalizations about decomposing using addition for any rational number.

- Ask students to repeatedly decompose a whole number using addition and then decompose a fraction using the same denominator and explain similarities. For example, decompose 428 as $400 + 20 + 8$ or $420 + 8$ or $200 + 200 + 10 + 10 + 4 + 4$ or another true expression. Then, decompose the fraction $\frac{17}{12}$ as $\frac{12}{12} + \frac{5}{12}$ or $1 + \frac{5}{12}$ or $\frac{10}{12} + \frac{7}{12}$ or another true expression. Make a generalization that both whole numbers and fractions can be decomposed by the units, whether they are hundreds, ones, or twelfths.

- Ask students to make generalizations about decomposition using addition for different values and test their generalizations. After decomposing a variety of whole numbers and fractions, ask students questions such as the following:
    - What do you notice about decomposing numbers?
    - What do you wonder about decomposing numbers?
    - When decomposing numbers, what is something that is always true, sometimes true, or never true, whether it is a whole number or a fraction?

  Accept any correct explanation with words, drawings, or manipulatives. Some correct explanations may be: "All numbers can be decomposed in many different ways." or "All whole numbers can be decomposed into fractions, but not all fractions can be decomposed into whole numbers."

How can addition and subtraction equations be used to convert between mixed numbers and fractions greater than 1?

M.P.2. Reason abstractly and quantitatively. Explain and show addition equations to find equivalence between mixed numbers and fractions greater than 1. For example, $\frac{8}{3} = \frac{3}{3} + \frac{3}{3} + \frac{2}{3} = 1 + 1 + \frac{2}{3} = 2\frac{2}{3}$ and vice versa. Additionally, connect that a mixed number is a whole number plus a fraction less than 1 without the addition symbol, e.g., $4\frac{1}{3}$ is the same as $4 + \frac{1}{3}$.

- Ask students to use addition of fractions with like denominators to decompose and compose mixed numbers and fractions greater than 1. Students should be able to use drawings, manipulatives, and expressions or equations to decompose a mixed number in a variety of ways, such as $3\frac{7}{10} = 3 + \frac{7}{10} = 1 + 1 + 1 + \frac{7}{10} = \frac{10}{10} + \frac{10}{10} + \frac{10}{10} + \frac{7}{10} = \frac{37}{10}$. Students should also be able to decompose a fraction greater than 1 into a mixed number, as shown in the diagram.

$$\frac{14}{3} = \boxed{\tfrac{1}{3}\,\big|\,\tfrac{1}{3}\,\big|\,\tfrac{1}{3}} + \boxed{\tfrac{1}{3}\,\big|\,\tfrac{1}{3}\,\big|\,\tfrac{1}{3}} + \boxed{\tfrac{1}{3}\,\big|\,\tfrac{1}{3}\,\big|\,\tfrac{1}{3}} + \boxed{\tfrac{1}{3}\,\big|\,\tfrac{1}{3}\,\big|\,\tfrac{1}{3}} + \boxed{\tfrac{1}{3}\,\big|\,\tfrac{1}{3}\,\big|\,\phantom{\tfrac{1}{3}}}$$

$$= \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{2}{3}$$

$$= 1 + 1 + 1 + 1 + \frac{2}{3}$$

$$= 4 + \frac{2}{3}$$

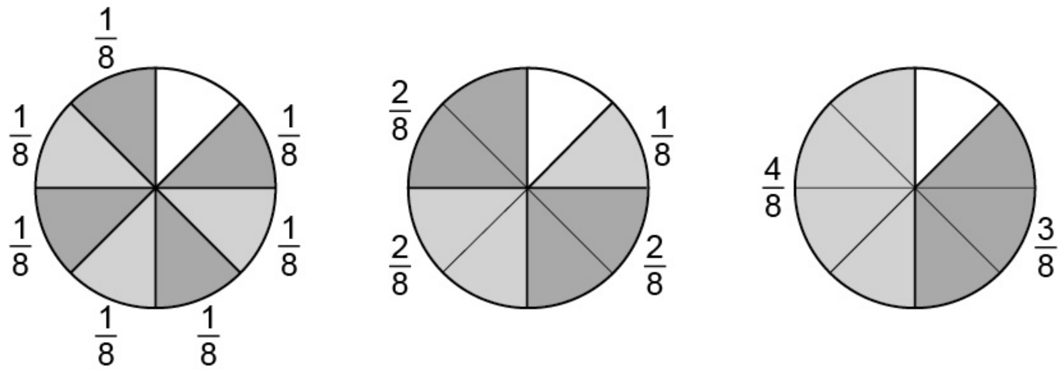
$$= 4\frac{2}{3}$$

- Ask students to write mixed numbers as a sum of a whole number and a fraction less than one, such as $2\frac{2}{3} = 2 + \frac{2}{3}$. Students should also read the numbers out loud to help reinforce this fact: "two and two-thirds is the same as two plus two-thirds."

How can drawings or visual models be used to justify the decomposition of a fraction?

M.P.4. Model with mathematics. Identify relationships between models and equations to justify decompositions. For example, use fraction manipulatives to show how many $\frac{1}{6}$ parts are in $\frac{5}{6}$, and then write an equation to represent the manipulatives. Additionally, recognize that a fraction can often be decomposed in multiple ways.

- Ask students to use manipulatives or draw models to justify a decomposition of a fraction. Students may use fraction circles to decompose $\frac{7}{8}$ in some of the ways shown.



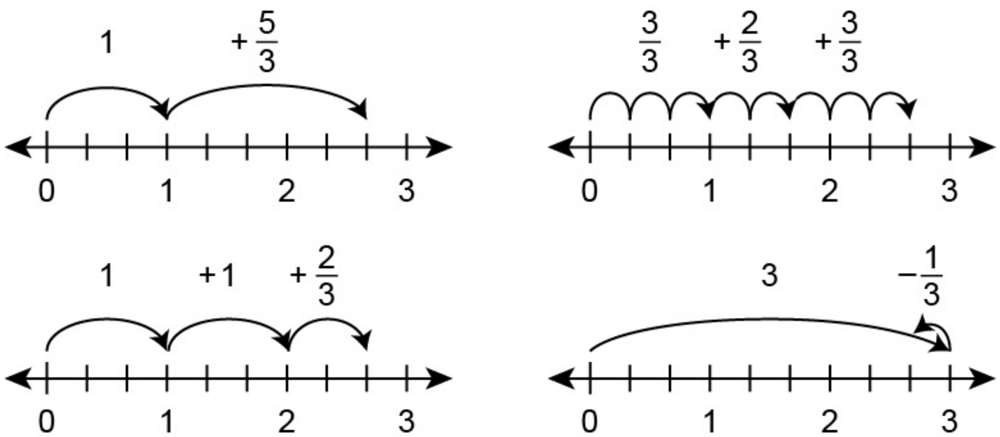
- Ask students to write equations to represent given models. For example, given the previous models, write $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$; $\frac{2}{8}+\frac{2}{8}+\frac{2}{8}+\frac{1}{8}$; and $\frac{4}{8}+\frac{3}{8}$.

How can any fraction, whole number, mixed number, or fraction greater than 1 be represented as the sum or difference of whole numbers and/or fractions?

M.P.2. Reason abstractly and quantitatively. Decompose fractions in a variety of ways. For example, $\frac{12}{5}$ can be decomposed in many different ways: $\frac{5}{5}+\frac{5}{5}+\frac{2}{5}$; $2+\frac{2}{5}$; $\frac{6}{5}+\frac{6}{5}$; $\frac{1}{5}+\frac{1}{5}+1+\frac{5}{5}$; $\frac{15}{5}-\frac{3}{5}$; etc. Additionally, move between a variety of forms of fractions fluently, using decompositions and models as justifications.

- Ask students to decompose and compose fractions using addition and subtraction by using expressions and equations. For example, decompose the number $1\frac{5}{3}$ in any number of ways, including, but not limited to: $1+\frac{5}{3}$, $\frac{3}{3}+\frac{2}{3}+\frac{3}{3}$, $1+1+\frac{2}{3}$, and $3-\frac{1}{3}$.

- Ask students to use models and equations to show equivalent decompositions of a value. For the value $1\frac{5}{3}$, students might draw diagrams to show the previous decompositions.



Key Academic Terms:

fractions, addition, subtraction, sum, difference, whole number, numerator, denominator, whole, equation, unit fraction, compose, mixed number, decompose

Additional Resources:

- Article: How to build fraction knowledge with a daily fraction printable
- Activity: Comparing sums of unit fractions
- Article: Here's an awesome way to teach kids fractions
- Activity: Making 22 seventeenths in different ways

15b

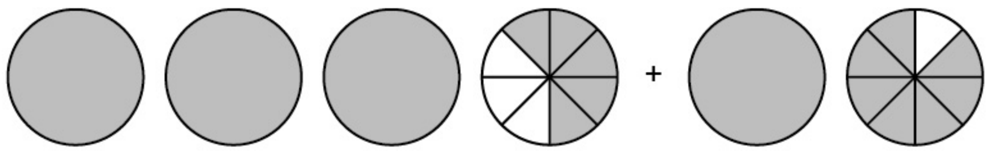
Operations with Numbers: Fractions
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
15. Model and justify decompositions of fractions and explain addition and subtraction of fractions as joining or separating parts referring to the same whole. b. Add and subtract fractions and mixed numbers with like denominators using fraction equivalence, properties of operations, and the relationship between addition and subtraction.

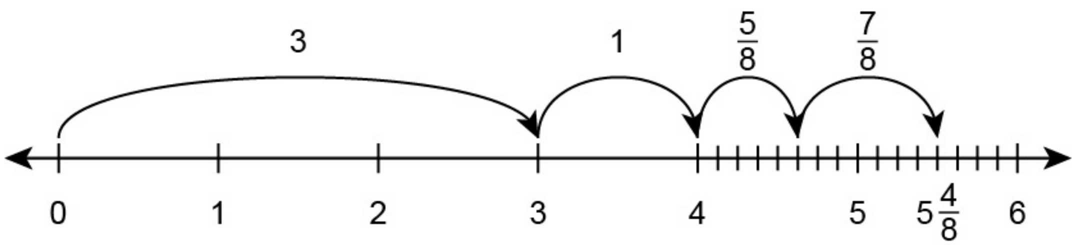
Guiding Questions with Connections to Mathematical Practices:

How can mixed numbers with like denominators be added or subtracted in a variety of ways?

M.P.1. Make sense of problems and persevere in solving them. Apply a variety of strategies to solve addition and subtraction problems with like denominators. For example, $3\frac{1}{6} - \frac{4}{6} = (2 + \frac{6}{6} + \frac{1}{6}) - \frac{4}{6} = 2\frac{7}{6} - \frac{4}{6} = 2\frac{3}{6}$. Additionally, recognize that the commutative and associative properties of adding whole numbers apply to fractions as well.

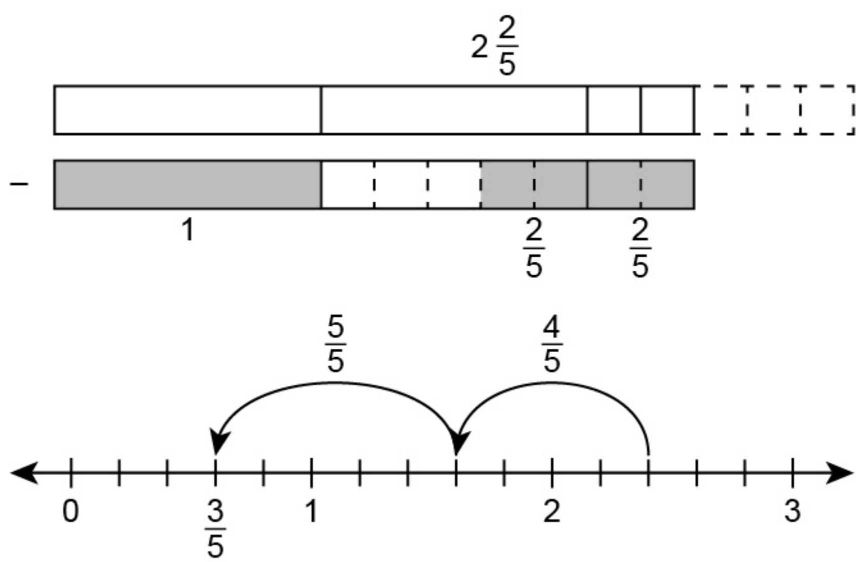
- Ask students to use visual models, including drawings and manipulatives, to add and subtract fractions and mixed numbers. For example, ask students to create a visual model for the problem $3\frac{5}{8} + 1\frac{7}{8}$. Some possible methods of finding the solution of $5\frac{4}{8}$ are shown.





$$4 + \frac{8}{8} + \frac{4}{8} = 4 + 1 + \frac{4}{8} = 5\frac{4}{8}$$

Further, ask students to model the problem $2\frac{2}{5} - 1\frac{4}{5}$. Some possible methods of finding the solution of $\frac{3}{5}$ are shown.



- Ask students to decompose fractions to write equivalent expressions and/or equations to add and subtract fractions and mixed numbers. When adding $6\frac{3}{12} + 3\frac{10}{12}$, students may decompose the numbers and use the properties of addition to write the following equivalent expressions.

$$6 + \frac{3}{12} + 3 + \frac{10}{12}$$

$$6 + 3 + \frac{3}{12} + \frac{10}{12}$$

$$6 + 3 + \frac{13}{12}$$

$$9 + \frac{13}{12}$$

$$9 + \frac{12}{12} + \frac{1}{12}$$

$$9 + 1 + \frac{1}{12}$$

$$10 + \frac{1}{12}$$

$$10\frac{1}{12}$$

Key Academic Terms:

fraction, addition, subtraction, mixed number, equivalent fraction, whole number, numerator, denominator, like denominators, decompose

Additional Resources:

- Video: Add mixed numbers with like denominators
- Activity: Cynthia's perfect punch

15c

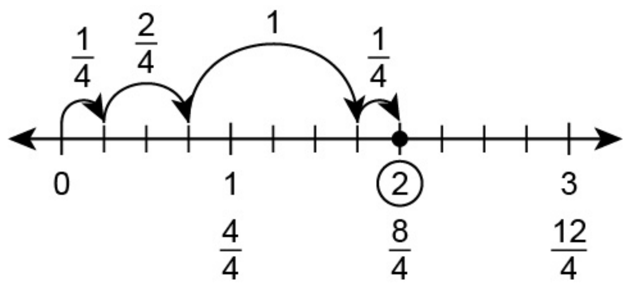
Operations with Numbers: Fractions
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
15. Model and justify decompositions of fractions and explain addition and subtraction of fractions as joining or separating parts referring to the same whole. c. Solve word problems involving addition and subtraction of fractions and mixed numbers having like denominators, using drawings, visual fraction models, and equations to represent the problem.

Guiding Questions with Connections to Mathematical Practices:

How can a fraction word problem be represented with an equation and visual model?

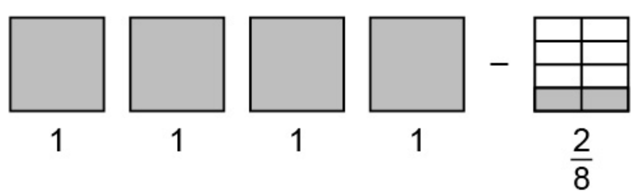
M.P.4. Model with mathematics. Explain connections between verbal descriptions, equations/expressions, and visual models of fraction word problems. For example, consider the following word problem: "Angel walked a total of $2\frac{1}{8}$ miles on Monday and Tuesday. Angel walked $\frac{5}{8}$ of a mile on Tuesday. How many miles did Angel walk on Monday?" This word problem connects to the expression $2\frac{1}{8} - \frac{5}{8}$. The expression is solved by using a fraction model, where one of the wholes is decomposed to use in subtraction. An equation is written to represent the fraction model: $1 + \frac{1}{8} + \frac{8}{8} - (\frac{4}{8} + \frac{1}{8}) = 1 + (\frac{1}{8} - \frac{1}{8}) + (\frac{8}{8} - \frac{4}{8}) = 1\frac{4}{8}$. The equation is then contextualized to state that Angel walked $1\frac{4}{8}$ miles on Monday. Additionally, given a sum or difference of two fractions, write a word problem that represents the expression.

- Ask students to represent a word problem with a visual or manipulative model. For example, give students the word problem "Michele reads for $\frac{1}{4}$ of an hour, plays outside for $1\frac{2}{4}$ hours, and eats a snack for $\frac{1}{4}$ of an hour. How many hours does Michele read, play outside, and eat all together?" One possible response is to draw the following model to find a solution of 2 hours.



- Ask students to represent a word problem with an equation or expression. For example, given the previous word problem, represent the problem as $\frac{1}{4} + 1\frac{2}{4} + \frac{1}{4} = 2$.

- Ask students to write a word problem for an equation, expression, or visual model. For example, give students the model shown.



A possible student response is "There are 4 cups of sugar in a jar. A recipe calls for $\frac{2}{8}$ cup sugar. How much sugar remains in the jar after using the $\frac{2}{8}$ cup of sugar?"

Key Academic Terms:

fraction, word problem, visual model, mixed number, whole number, numerator, denominator, like denominators, expression, equation, decompose

Additional Resources:

- Lesson: Grade 4 Mathematics module 5, topic D, lesson 19
- Video: Subtraction of fractions using a visual model

16a

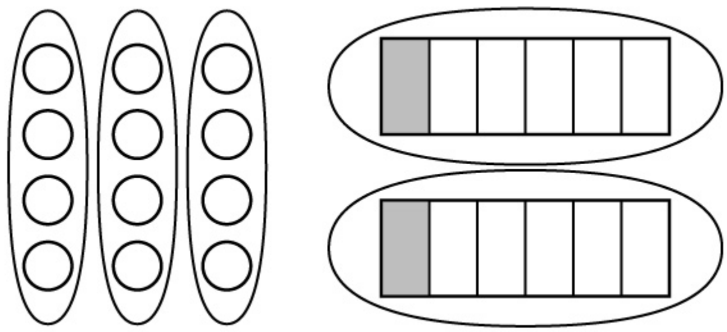
Operations with Numbers: Fractions
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
16. Apply and extend previous understandings of multiplication to multiply a whole number times a fraction. a. Model and explain how a non-unit fraction can be represented by a whole number times the unit fraction. *Example*: $\frac{9}{8} = 9 \times \frac{1}{8}$

Guiding Questions with Connections to Mathematical Practices:

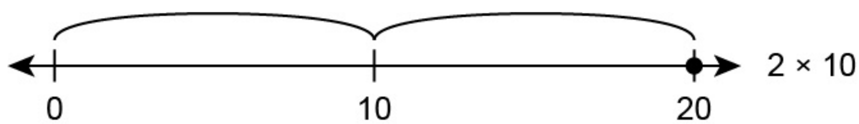
How is multiplying a unit fraction by a whole number similar to and different from multiplying two whole numbers?

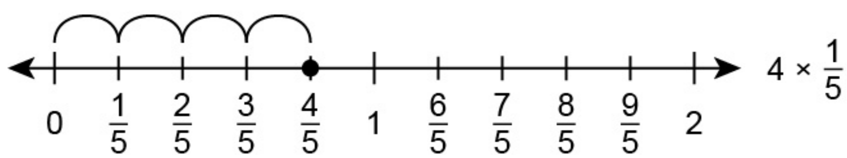
M.P.7. Look for and make use of structure. Explain the similarities and differences between multiplication of whole numbers and multiplication of a unit fraction by a whole number. For example, the meaning of multiplication stays the same; multiplication is always represented by either a multiplicative comparison or "groups of" numbers which represent repeated addition. Additionally, when multiplying positive whole numbers, the product is always greater than or equal to both factors, but when multiplying a unit fraction by a positive whole number, the product is less than or equal to the whole number but greater than or equal to the unit fraction.

- Ask students to model and explain multiple ways in which multiplying a unit fraction by a whole number is similar to multiplying two whole numbers. Use arrays, verbal descriptions, or other visual representations like number lines.



3×4 is 3 groups of 4 just like $2 \times \frac{1}{6}$ is 2 groups of $\frac{1}{6}$



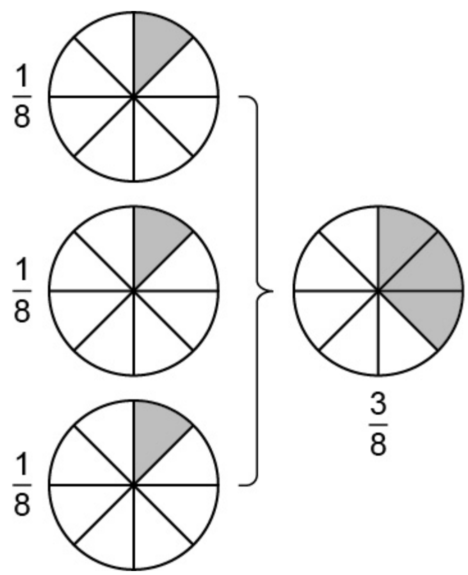


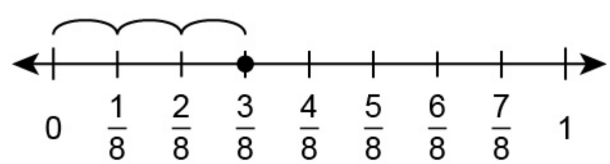
- Ask students to model and explain multiple ways in which multiplying a unit fraction by a whole number is different from multiplying two whole numbers. Use verbal justifications, such as "multiplying whole numbers will always have a product that is a whole number, but multiplying a whole number and a unit fraction can have a product that is a whole number or a multiple of a unit fraction," and support the conclusion with equations such as $2 \times 3 = 6$, $1 \times 1 = 1$, $4 \times \frac{1}{10} = \frac{4}{10}$, and $8 \times \frac{1}{2} = \frac{8}{2} = 4$.

How can a fraction be decomposed and composed into a product of a whole number times a unit fraction?

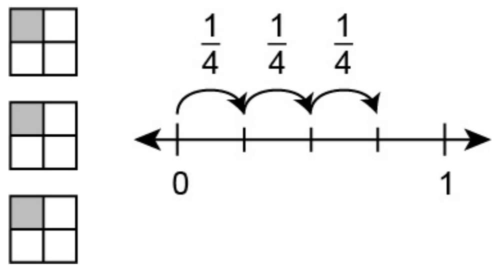
M.P.5. Use appropriate tools strategically. Use visual models in a variety of ways to show how to multiply a unit fraction by a whole number. For example, demonstrate multiplication of $\frac{1}{8} \times 3$ using methods such as skip-counting on a number line (e.g., "one eighth, two eighths, three eighths"), or drawing pictures of $\frac{1}{8}$ three times. Additionally, describe multiplication in a variety of ways, verbally and in writing, such as describing $7 \times \frac{1}{2}$ as "seven times one-half," "7 groups of one-half," and "seven halves." Further, rewrite the expression as $\frac{1}{2} \times 7$ and say "half of seven."

- Ask students to write fractions as the product of a unit fraction and a whole number and to represent their product with manipulatives or visual models. For example, give students the fraction $\frac{3}{8}$. The fraction can be written as the product $\frac{1}{8} \times 3$. Two possible models are shown.





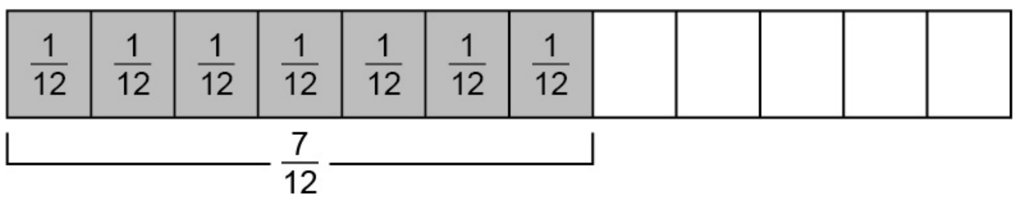
- Ask students to use concrete objects and visual models to represent multiplication of a unit fraction and a whole number, no matter the order of the values. For example, ask students to model $\frac{1}{4} \times 3$ using an area fraction model or a number line to find a solution of $\frac{3}{4}$. Two possible models are shown.

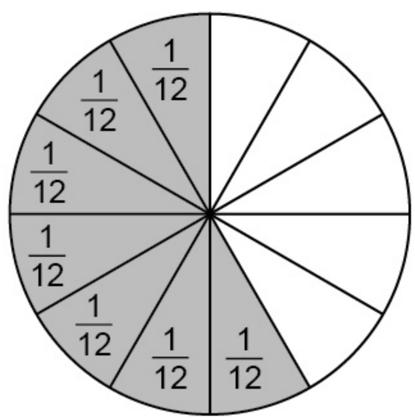


- Ask students to use multiplication language to describe expressions, equations, and visual representations of multiplication of a whole number and a unit fraction. Describe an expression or visual representation, such as the previous expression and visual models for $\frac{1}{4} \times 3$, using phrases such as "one-fourth of three," "three groups of one-fourth," or "three-fourths is three times as large as one-fourth."

M.P.2. Reason abstractly and quantitatively. Attend to the meaning of a quantity and decompose fractions using visual models to write equations with repeated addition and show the equivalent multiplication equation. For example, $\frac{4}{3}$ can be decomposed and written as $\frac{4}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 4 \times \frac{1}{3}$. Additionally, model $\frac{4}{3}$ by doing 4 jumps of $\frac{1}{3}$ on a number line to illustrate the action of multiplication.

- Ask students to represent non-unit fractions using unit fractions with visual models. For example, represent the fraction $\frac{7}{12}$ in a variety of ways, two of which are shown.





- Ask students to represent non-unit fractions using equations involving unit fractions. Given the fraction $\frac{7}{12}$ and using models, write multiple equivalent equations, such as $\frac{7}{12} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = 7 \times \frac{1}{12} = 3 \times \frac{1}{12} + 4 \times \frac{1}{12}$.

How does the fraction $\frac{a}{b}$ compare with the fraction $\frac{1}{b}$?

M.P.8. Look for and express regularity in repeated reasoning. Identify the generalization that $\frac{a}{b}$ is a multiple of $\frac{1}{b}$ after numerous experiences decomposing various fractions and mixed numbers. For example, given the fraction $\frac{6}{8}$, immediately recognize that it can be decomposed into $6 \times \frac{1}{8}$. Additionally, use multiplicative comparison to explain that $\frac{6}{8}$ is 6 times as great as $\frac{1}{8}$.

- Ask students to make generalizations about the relationship between a fraction and the unit fraction with the same denominator. Students may form conclusions such as "the numerator is the number of unit fractions that make up the fraction" and "since the unit fraction is repeatedly added, use multiplication of a whole number and the unit fraction to write an equation to compare the fraction and the unit fraction."
- Ask students to write multiplication expressions and/or equations of a whole number and unit fraction for any fraction $\frac{a}{b}$. Students should fluently write multiplication equations for fractions such as $\frac{3}{8} = 3 \times \frac{1}{8}$ or $\frac{9}{2} = 9 \times \frac{1}{2}$.

Key Academic Terms:

unit fraction, numerator, denominator, equation, product, multiple, multiply, whole number, mixed number, decompose, expression, multiplicative comparison, generalization, concrete object, visual representation

Additional Resources:

- Article: Multiplying fraction activity to teach whole numbers by fractions
- Activity: Extending multiplication from whole numbers to fractions

16b

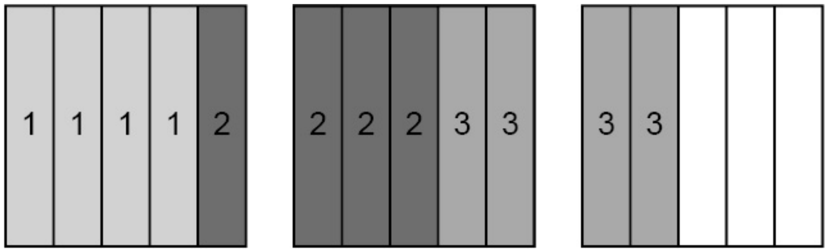
Operations with Numbers: Fractions
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
16. Apply and extend previous understandings of multiplication to multiply a whole number times a fraction. b. Extend previous understanding of multiplication to multiply a whole number times any fraction less than one. *Example*: $4 \times \frac{2}{3} = \frac{4 \times 2}{3} = \frac{8}{3}$

Guiding Questions with Connections to Mathematical Practices:

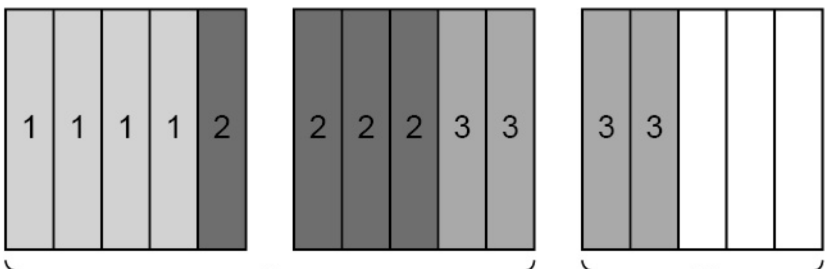
How can fractions be multiplied by whole numbers in a variety of ways?

M.P.5. Use appropriate tools strategically. Use visual models, equations, and the properties of operations to decompose and compose numbers to solve the multiplication of a fraction and a whole number. For example, solve $\frac{7}{3} \times 2$ by decomposing $\frac{7}{3} = 7 \times \frac{1}{3}$ and then multiplying $2 \times 7 \times \frac{1}{3}$ to find $14 \times \frac{1}{3} = \frac{14}{3}$. Additionally, decompose fractions using multiplication, such as $\frac{6}{8} = 6 \times \frac{1}{8} = 3 \times 2 \times \frac{1}{8} = 3 \times \frac{2}{8}$ or any equivalent expression.

- Ask students to decompose fractions and whole numbers using visuals and expressions to multiply. For example, visually represent $\frac{4}{5} \times 3$ as shown.



Write expressions or equations that could represent the solution, such as $\frac{4}{5} + \frac{4}{5} + \frac{4}{5}$, $\frac{3 \times 4}{5}$, $4 \times \frac{3}{5}$, or $2 \times \frac{5}{5} + 1 \times \frac{2}{5}$. Use visuals to show the equation, such as the example shown.



$$2 \times \frac{5}{5} + 1 \times \frac{2}{5}$$

$$2 \times 1 + 1 \times \frac{2}{5}$$

- Ask students to use the properties of operations and the meaning of multiplication to multiply whole numbers and fractions. Students should write equivalent expressions to multiply whole numbers and fractions. Students do not need to name these properties but need to use the properties fluently while decomposing and composing. Students could write a variety of equivalent expressions to $\frac{5}{6} \times 4$, including $4 \times \frac{5}{6}$, $\frac{4 \times 5}{6}$, $\frac{5}{6} + \frac{5}{6} + \frac{5}{6} + \frac{5}{6}$, and $2 \times \frac{5}{6} + 2 \times \frac{5}{6}$. Reminder: Students do not need to know the names of the properties of operations.

M.P.8. Look for and express regularity in repeated reasoning. Notice the generalization, after many examples of multiplying a whole number by a fraction, that $n \times (\frac{a}{b}) = (\frac{n \times a}{b})$. For example, given the problem $8 \times \frac{2}{3}$, immediately know that it can be solved as $8 \times \frac{2}{3} = \frac{16}{3}$. Additionally, justify the generalization by saying "$8 \times \frac{2}{3}$ is 8 jumps of $\frac{2}{3}$ on a number line, and since each jump is a group of 2 one-thirds, there are 8×2 jumps of $\frac{1}{3}$, which is 16 one-thirds."

- Ask students to generalize a rule for multiplying any fraction by a whole number. Students should state something similar to "when multiplying a whole number and a fraction, the denominator stays the same and the numerator and whole number are multiplied."
- Ask students to justify the rule for multiplying a fraction and a whole number using words, pictures, and/or equations. For example, use the equation $2 \times \frac{5}{2} = 2 \times 5 \times \frac{1}{2} = 10 \times \frac{1}{2} = \frac{10}{2} = \frac{2 \times 5}{2}$ to represent a case of the generalization that $n \times (\frac{a}{b}) = (\frac{n \times a}{b})$.

Key Academic Terms:

unit fraction, numerator, denominator, equation, product, multiple, multiply, whole number, mixed number, compose, decompose, properties of operation, expression, generalization

Additional Resources:

- Article: Multiplying fractions
- Activity: First to 50 (fractions of groups game)

16c

Operations with Numbers: Fractions
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
16. Apply and extend previous understandings of multiplication to multiply a whole number times a fraction. c. Solve word problems involving multiplying a whole number times a fraction using visual fraction models and equations to represent the problem. *Examples*: $3 \times \frac{1}{2}$, $6 \times \frac{1}{8}$

Guiding Questions with Connections to Mathematical Practices:

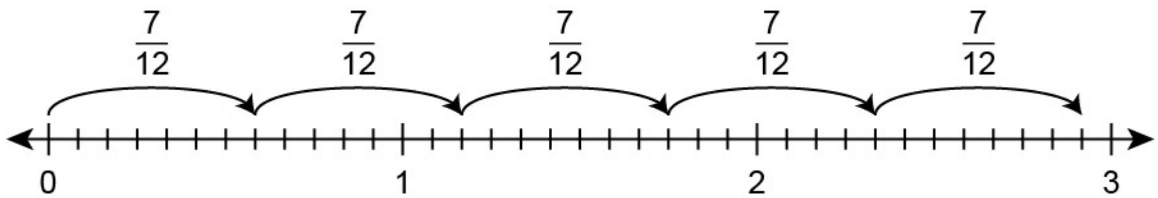
How can a fraction word problem be represented with an equation and visual model?

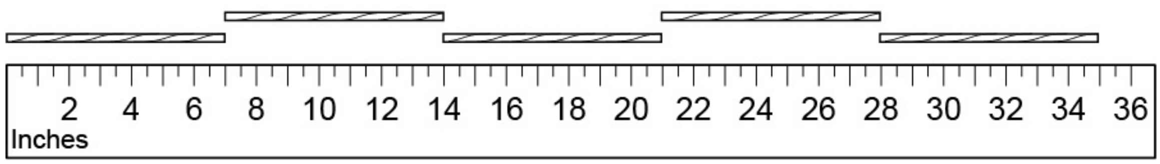
M.P.1. Make sense of problems and persevere in solving them. Explain connections between verbal descriptions, equations/expressions, and visual models of fraction word problems. For example, consider the following word problem: "A recipe calls for $1\frac{3}{4}$ cups of flour. Javier triples the recipe. How much flour does Javier use?" This word problem connects to the expression $1\frac{3}{4} \times 3$. The expression could be represented by drawing $1\frac{3}{4}$ rectangles three times. An equation could also be written to represent the model:

$$1\frac{3}{4} \times 3 = (1 + 1 + 1) + (\frac{3}{4} + \frac{3}{4} + \frac{3}{4}) = (3 \times 1) + (3 \times \frac{3}{4}) = 3 + \frac{9}{4} = 3 + \frac{4}{4} + \frac{4}{4} + \frac{1}{4} = 5\frac{1}{4}$$

The equation is then contextualized to state that Javier uses $5\frac{1}{4}$ cups of flour. Additionally, explicit connections between the representations can be made, such as "there are 3 groups of $1\frac{3}{4}$ rectangles, which is represented by $1\frac{3}{4} \times 3$. There are 3 wholes and 3 three-fourths rectangles, which is represented as $(3 \times 1) + (3 \times \frac{3}{4})$.

- Ask students to solve word problems by creating visual representations of the problem using manipulatives and/or drawings. For example, considering the word problem, "Teresa is making 5 bracelets for her friends. Each bracelet uses $\frac{7}{12}$ foot of fabric. How many feet of fabric does Teresa use making the bracelets?" draw a representation using a number line or ruler as shown.





- Ask students to solve word problems by writing an equation to represent the word problem. For example, after students represent the previous problem visually, they should write an equation such as $5 \times \frac{7}{12} = \frac{5 \times 7}{12} = \frac{35}{12}$.

Key Academic Terms:

unit fraction, numerator, denominator, equation, product, multiple, multiply, whole number, mixed number, decompose, visual representation

Additional Resources:

- Article: [Multiplying fractions and whole numbers: two types of problems]
- Activity: [Sugar in six cans of soda]

17a

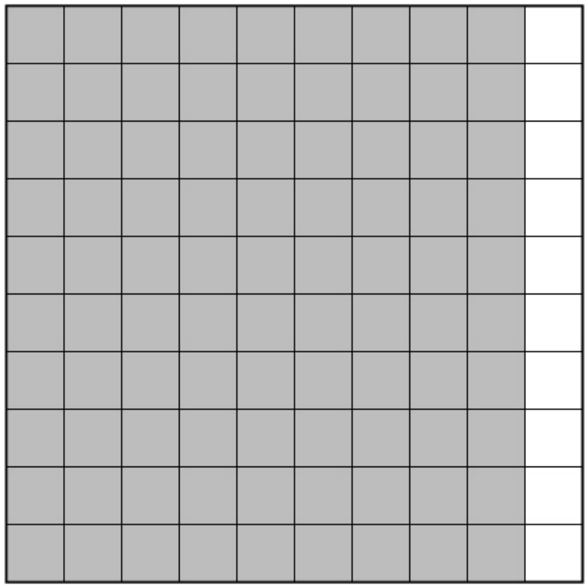
Operations with Numbers: Fractions
Understand decimal notation for fractions, and compare decimal fractions. *Denominators are limited to 10 and 100.*
17. Express, model, and explain the equivalence between fractions with denominators of 10 and 100. a. Use fraction equivalency to add two fractions with denominators of 10 and 100.

Guiding Questions with Connections to Mathematical Practices:

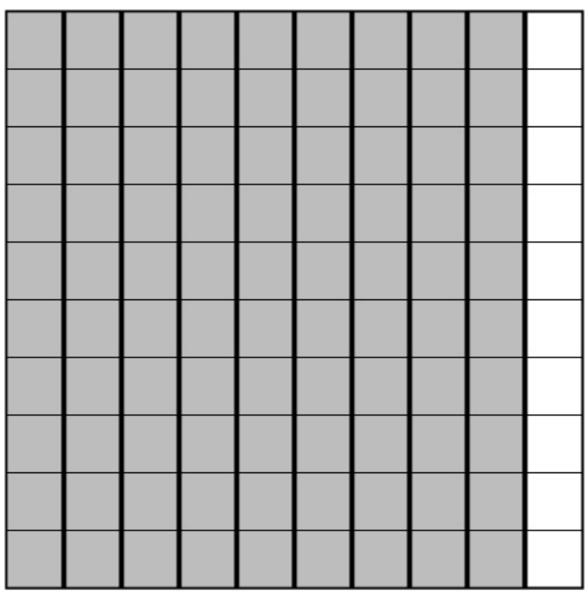
How can equivalent fractions be created for fractions with 10 and 100 in the denominator?

M.P.5. Use appropriate tools strategically. Convert a fraction with denominator 10 to an equivalent fraction of denominator 100 by using fraction models such as base-ten blocks or hundredths grids. For example, represent $\frac{2}{10}$ using two long base-ten blocks and demonstrate that the quantity is equivalent to 20 small base-ten cubes. Additionally, represent $\frac{80}{100}$ on a hundredths grid and note that it is equivalent to $\frac{8}{10}$.

- Ask students to use models to represent equivalent fractions with 10 or 100 in the denominator. For example, the diagram shows $\frac{90}{100}$.



When distinguishing tenths, identify that an equivalent fraction is $\frac{9}{10}$, as shown.

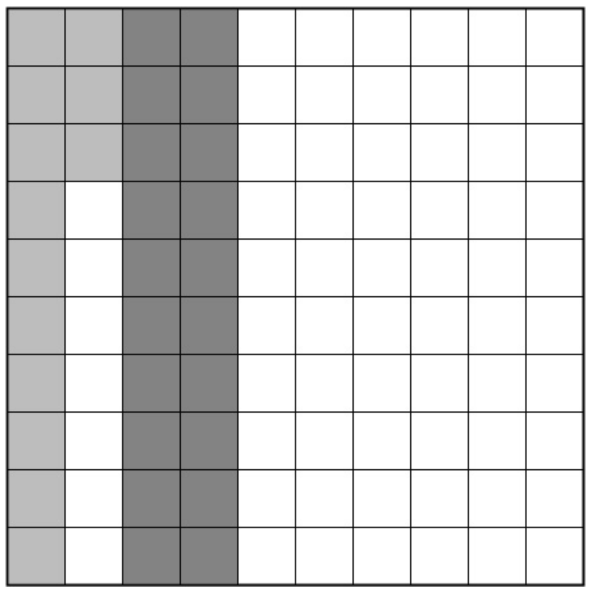


- Ask students to generalize how to generate equivalent fractions with 10 or 100 in the denominator. Students should notice that multiplying or dividing by $\frac{10}{10}$ generates an equivalent fraction with 10 or 100 in the denominator. For example, to find the equivalent fraction with a denominator of 100 for the fraction $\frac{7}{10}$, write $\frac{7}{10} \times \frac{10}{10} = \frac{70}{100}$.

How can two fractions with denominators 10 and 100 be added?

M.P.4. Model with mathematics. Apply more than one strategy for adding tenths and hundredths. For example, to add $\frac{4}{10}+\frac{11}{100}$, use base-ten blocks to show $\frac{4}{10}=\frac{40}{100}$ and add $\frac{40}{100}+\frac{11}{100}=\frac{51}{100}$. Then connect the visual representation with an equation. Additionally, find equivalent fractions using multiplication to add and subtract fractions with denominators of 10 and 100.

- Ask students to use models to add fractions with 10 or 100 in the denominator. For example, the diagram represents $\frac{13}{100}+\frac{2}{10}$ for a solution of $\frac{33}{100}$.



- Ask students to solve addition problems involving fractions with 10 and 100 in the denominator by using equations and multiplication to find equivalent fractions. For example, to solve $\frac{28}{100}+\frac{4}{10}$, write $\frac{4}{10}\times\frac{10}{10}=\frac{40}{100}$ and then solve $\frac{28}{100}+\frac{40}{100}=\frac{68}{100}$.

Key Academic Terms:

fractions, fraction models, tenths, hundredths, numerator, denominator, addition, hundredths grid, equivalent fraction, equivalency

Additional Resources:

- Video: Equivalent fractions with denominators 10 and 100
- Video: Fractions with a denominator of 10 and 100

18

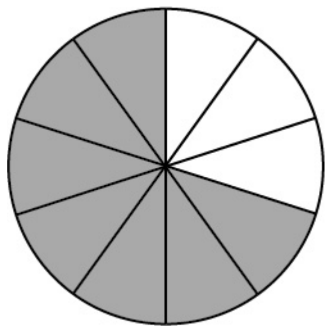
Operations with Numbers: Fractions
Understand decimal notation for fractions, and compare decimal fractions. *Denominators are limited to 10 and 100.*
18. Use models and decimal notation to represent fractions with denominators of 10 and 100.

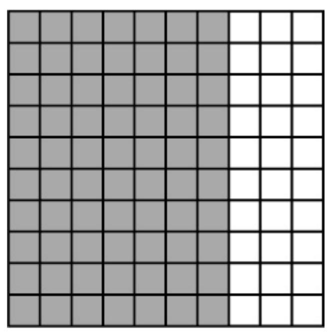
Guiding Questions with Connections to Mathematical Practices:

How can fractions with denominators of 10 or 100 be written in decimal notation?

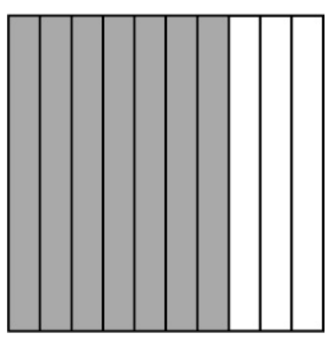
M.P.8. Look for and express regularity in repeated reasoning. Apply place value patterns for relating fractions with denominators of 10 and 100 to decimals. For example, notice that tenths written as fractions have a single zero in the denominator and that tenths written as decimals are located a single space to the right of the decimal point (e.g., 2.8 has one digit after the decimal point, so it is written in fraction form as $2\frac{8}{10}$). Additionally, use place value understanding to write decimal numbers greater than 1 in a variety of ways, such as $1.6 = 1.60 = \frac{160}{100} = \frac{16}{10}$.

- Ask students to write fractions as decimals and decimals as fractions in many different equivalent ways. For example, write the fraction $\frac{7}{10}$ as $\frac{70}{100}$ or 0.7 or 0.70 or .7 or .70, etc. Represent fractions and decimals with models and diagrams, like the drawings shown.





$$\frac{70}{100}$$



$$\frac{7}{10}$$

- Ask students to generalize a rule about how to write equivalent fractions and decimals. For example, "When a number has one digit after the decimal point, it is read as and written in tenths. When a number has two digits after the decimal point, it is read as and written in hundredths."

How does place value language extend to decimals to describe decimals in a variety of ways?

M.P.6. Attend to precision. Connect whole number place value language to decimals and use place value language to describe decimals in a variety of ways. For example, describe 0.13 as "one tenth and three hundredths" or "thirteen hundredths." Additionally, decompose fractions written in tenths and hundredths using multiplication of a whole number and a fraction to reinforce place value language and decimal understanding.

- Ask students to write and describe decimals and fractions in a variety of equivalent ways including using different place values as the unit. For example, describe the number 3.2 as "three and two-tenths," "thirty-two tenths," or "three hundred twenty hundredths." Write the decimal as a variety of equivalent fractions and mixed numbers to reinforce place value understanding, such as $3\frac{2}{10}$, $\frac{32}{10}$, and $\frac{320}{100}$.
- Ask students to decompose decimals to reinforce place value understanding. For example, write 3.2 as $3 \times 1 + 2 \times \frac{1}{10}$ or $32 \times \frac{1}{10}$ and read it as "three wholes and two one-tenths" and "thirty-two one-tenths."

How are decimals used in the real world?

M.P.2. Reason abstractly and quantitatively. Represent decimals using symbols on paper and explain what they mean. For example, an amount of money (35 cents), a distance (35 hundredths of a kilometer), and an amount of time (35 hundredths of a second faster than another runner) all can be represented on paper as 0.35. Additionally, differentiate between decimals used for mathematics and periods used to represent situations such as the version number of a software application or the date.

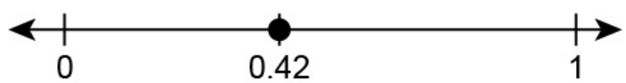
- Ask students to notice where decimals or decimal language is used in everyday life. For example, note weight on a scale, prices at a store, the time of a race, numbers at the gas station (both the cost and the number of gallons), distance on directions from a map website, or numbers on a calculator.

- Ask students to write realistic situations in which decimals are used. For example, write situations such as "Juan put 11.34 gallons of gas in the car" or "A bag of chips costs $0.99." As students become more familiar with quantity, it is expected that they are accurate in their approximations; students should not write situations such as "Jackie ran 12 miles in 0.01 seconds" as this is not reasonable.

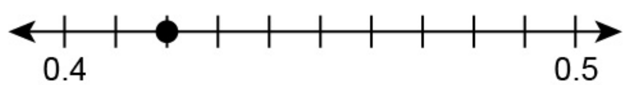
How can decimals be located on a number line?

M.P.5. Use appropriate tools strategically. Locate decimals on a number line. For example, demonstrate how a meter stick can be used as a number line, representing a whole of 100 centimeters or 1 meter, to show that 16 centimeters is equivalent to 16 hundredths of a meter (as well as 1 tenth and 6 hundredths of a meter) and can be written as 0.16 or $\frac{16}{100}$. Additionally, create number lines and double number lines that extend past 1 to make sense of decimal numbers and fractions.

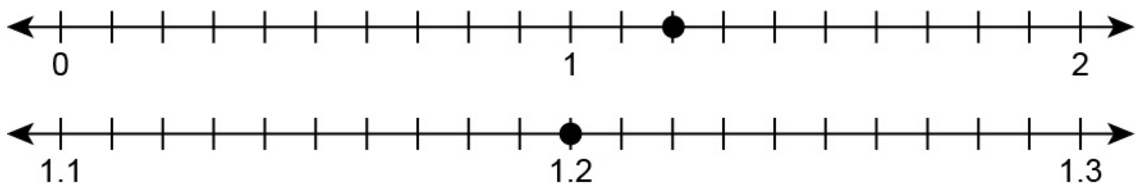
- Ask students to place decimals on number lines, both on precise and approximate locations, and explain their reasoning. For example, plot the decimal 0.42 in the ways shown.



Students may explain their reasoning as "0.42 is a little less than one-half, so it will be close to the middle, but closer to zero than one."



- Ask students to create number lines and plot decimals on them. For example, create multiple number lines that represent the decimal 1.2 in two different ways. Remind students to read the labels on a number line. Even though the number lines shown are the same length, they have different start and end points. Also, the tick marks on the first number line represent intervals of tenths, and the tick marks on the second number line represent intervals on hundredths.



Note that as students are learning decimals, it is a common error for students to place 0.10 after 0.9 instead of 1 when counting by tenths on the number line. Use manipulatives such as base-ten blocks or hundredths grids to reinforce the idea that after 0.9 is 1 whole, not 1 tenth.

Key Academic Terms:

Additional Resources:

- Activity: Three-way memory (decimals practice)
- Lesson: Coloring tiles—decimal designs
- Article: Hands-on activities for decimal tenths and hundredths
- Article: How to introduce decimals with base-ten blocks

19

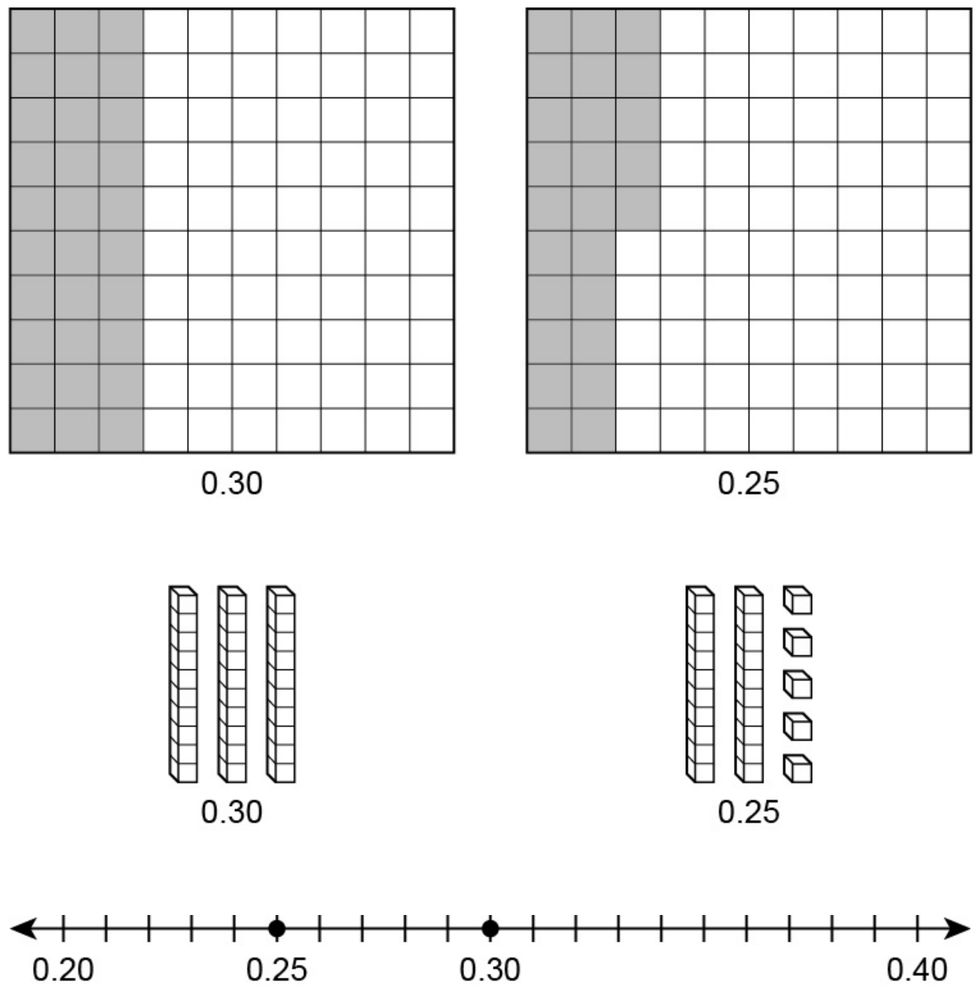
Operations with Numbers: Fractions
Understand decimal notation for fractions, and compare decimal fractions. *Denominators are limited to 10 and 100.*
19. Use visual models and reasoning to compare two decimals to hundredths (referring to the same whole), recording comparisons using symbols >, =, or <, and justifying the conclusions.

Guiding Questions with Connections to Mathematical Practices:

What strategies can be used to determine the size of a decimal and compare it to another decimal?

M.P.5. Use appropriate tools strategically. Extend place value understanding from whole numbers to demonstrate how to compare decimals that are in the tenths or hundredths place by using models such as base-ten blocks, hundredth grids, number lines, or equivalent number forms. For example, show that 0.2 is greater than 0.11 by shading each value on a hundredths grid. Additionally, place 0.2 further to the right on the number line than 0.11.

- Ask students to use manipulatives to compare decimals. For example, when comparing 0.3 and 0.25, students may represent the comparison as shown.

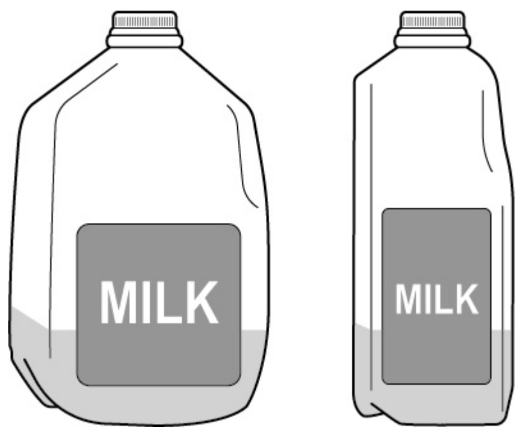


- Ask students to make informal justifications and generalizations about comparing decimals. Generate a variety of true statements such as “when comparing decimals with a zero in the ones place, it doesn’t matter the number of digits after the decimal point. You first compare the tenths place and the larger value means that that entire number is greater” or “numbers farther to the right on the number line are larger than any numbers that are to the left of the number.”

When is it appropriate to compare two decimals?

M.P.2. Reason abstractly and quantitatively. Know that when comparing decimals, the decimals must refer to the same whole. For example, if a hundredths grid is 10 centimeters by 10 centimeters and another hundredths grid is 10 inches by 10 inches, the wholes are not the same size, so it would not be appropriate to compare decimals related to the two different-sized grids. Additionally, if two decimal numbers are given without indication of the whole, it is assumed the whole is the same and the decimals can be compared.

- Ask students to write contexts and draw pictures to show when decimals should or should not be compared. Give students the scenario "0.25 of half a gallon of milk is less than 0.25 of a full gallon of milk, since they are not comparing to the same whole" along with a drawing to show that they are not the same quantity.



Students should connect that, in this respect, comparing decimals is similar to comparing other numbers.

- Ask students to make justifications about the meaning of decimals and the importance of the whole when comparing, whether the decimals are equal or unequal. For example, when comparing decimals without context, such as 1.2 and 0.82, the whole is assumed to be the same, so 1.2 is greater than 0.82. When given a context, whether it is a story or drawing, it is essential to ensure that the wholes are the same before comparing the decimals to ensure the comparison is accurate. Then the decimals can be compared.

How can two decimals be compared using symbols?

M.P.4. Model with mathematics. Record the results of the comparison of two decimals by using the mathematical symbols >, <, or =. For example, $0.6 > 0.44$ and $0.70 = 0.7$.

- Ask students to recall the use of comparison symbols used with whole numbers and fractions in previous grades. Observe that since the symbols >, =, or < are used when comparing whole numbers and fractions to make true statements, the symbols are also used to compare decimals, such as $0.78 > 0.62$.

- Ask students to write a missing decimal that is greater than, less than, or equal to another decimal using symbols; find multiple solutions to expressions such as $\square < 0.3$ (such as 0.1, 0.25, 0.01, or many other decimals); or write true comparisons with decimals, such as $0.90 > 0.89$.

Key Academic Terms:

compare, greater than, less than, equals, tenths, hundredths, whole, hundredths grid, visual representation, base ten, decimal, decimal point

Additional Resources:

- Lesson: Comparing and ordering decimals using money
- Lesson: Comparing decimals
- Lesson: The depth of decimals: comparing using a fractional model

20a

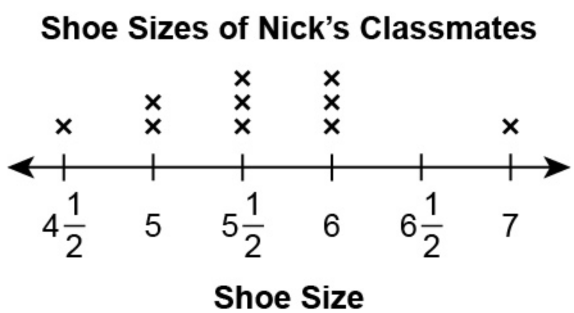
Data Analysis
Represent and interpret data.
20. Interpret data in graphs (picture, bar, and line plots) to solve problems using numbers and operations. a. Create a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$).

Guiding Questions with Connections to Mathematical Practices:

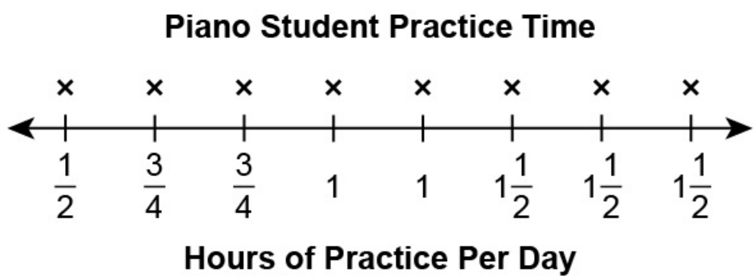
How is a line plot used to display a set of data?

M.P.6. Attend to precision. Create a line plot to represent measurement data, paying attention to the key features of a line plot. For example, measure and record the amount of precipitation for one month and then represent the collected data by creating a line plot using the correct scale, title, and label. Additionally, know that each point plotted on a line plot represents one value from the data set, and if a value on a line plot contains multiple points, then there are multiple instances of that value in the data set.

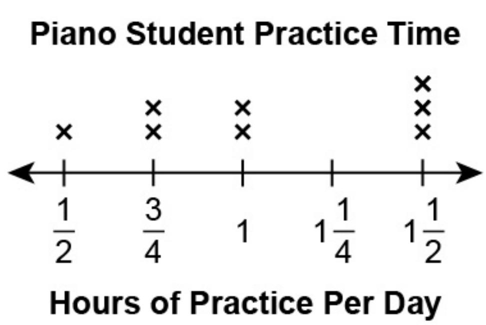
- Ask students to create a line plot to represent measurement data. For example, give students the prompt "Nick recorded the shoe sizes of ten classmates so he could order shoes for a performance. The sizes are shown in the following list: $4\frac{1}{2}$, 6, 6, $5\frac{1}{2}$, 5, $5\frac{1}{2}$, 6, 5, $5\frac{1}{2}$, and 7. Create a line plot to represent the data." Determine that the smallest size given is $4\frac{1}{2}$ and the largest size given is 7, so the line plot needs to contain values ranging from $4\frac{1}{2}$ to 7. Explain that the shoe sizes given are all either whole numbers or values halfway between two whole numbers, so the most appropriate scale for the line plot is $\frac{1}{2}$. The title of the line plot should describe the data in the context of the situation, so the title "Shoe Sizes of Nick's Classmates" can be used. The label below the line plot should indicate the units of measurement; in this case, the units measured by the line plot are shoe sizes, so the label "Shoe Size" can be used. The units measured by the line plot are shoe sizes, so the line plot needs to reflect that. Create the line plot using the specified parameters and plot each data point on the line plot.



- Ask students to analyze errors made in the construction of a line plot and to correct the errors to create a line plot with an appropriate scale, title, and label. For example, give students the prompt "Abby asked eight students who take piano lessons how much time, in hours, they spend practicing each day to prepare for a recital. The results were $\frac{1}{2}$ hour, $\frac{3}{4}$ hour, $1\frac{1}{2}$ hours, 1 hour, $1\frac{1}{2}$ hours, 1 hour, $\frac{3}{4}$ hour, and $1\frac{1}{2}$ hours. Abby displayed the data on a line plot.



Explain the error Abby made, and create a correct line plot for Abby's data." Determine that Abby made errors involving the scale of the line plot. Abby used a unique location on the line plot for each data point, even though some data points have the same value. Also, Abby did not include $1\frac{1}{4}$ hours on the line plot despite using a scale of $\frac{1}{4}$ hour, likely because there are no data points for that value. Explain that the correct line plot should contain values ranging from $\frac{1}{2}$ hour to $1\frac{1}{2}$ hours using a scale of $\frac{1}{4}$ hour, where each value has only one location on the line plot. Construct a line plot using the same title and label that Abby used with the corrections described above.



Key Academic Terms:

line plot, measurement, fraction, mixed number, data set, visual model, maximum, minimum, scale, title, label, analyze, interpret

Additional Resources:

- Video: Fractions on a line plot
- Article: Line plot activities and resources
- Activity: Aluminum foil boats

20b

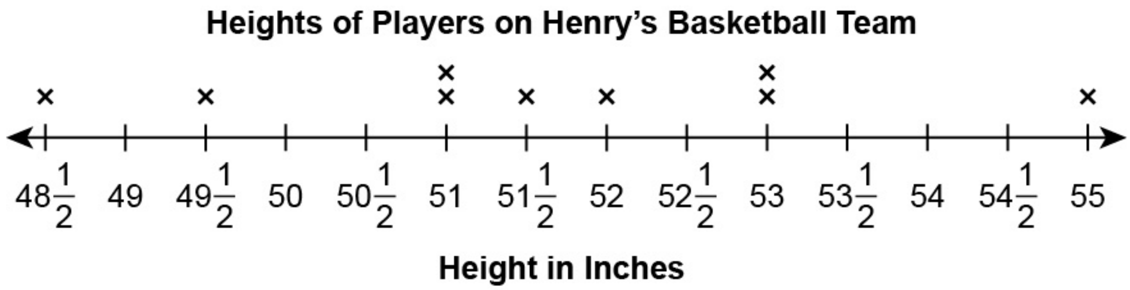
Data Analysis
Represent and interpret data.
20. Interpret data in graphs (picture, bar, and line plots) to solve problems using numbers and operations. b. Solve problems involving addition and subtraction of fractions using information presented in line plots.

Guiding Questions with Connections to Mathematical Practices:

How can the data in a line plot be analyzed to solve real-world problems?

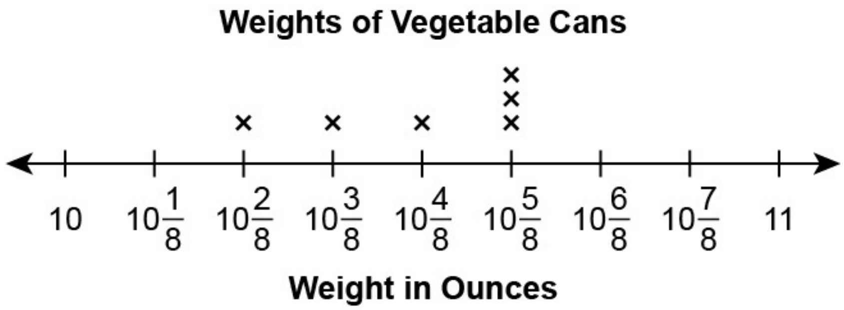
M.P.2. Reason abstractly and quantitatively. Analyze a line plot in context to solve addition and subtraction problems. For example, find the difference between the minimum and maximum values of a data set representing the hand lengths of the fourth graders in a classroom if the minimum value is the shortest hand length and the maximum value is the longest hand length. Additionally, know that each data point on a line plot represents a value and the values can be added together to find the sum of the data.

- Ask students to analyze a given line plot to solve a problem in the context of the situation the line plot represents. For example, give students the prompt "Henry measured the heights of the nine players on his basketball team. He used a line plot to represent the data.



What is the difference in height, in inches, between the tallest player and the shortest player on Henry's basketball team?" Determine that the point farthest to the left on the line plot represents the shortest player, who is $48\frac{1}{2}$ inches tall. The point farthest to the right on the line plot represents the tallest player, who is 55 inches tall. The problem asks for the difference between these heights; therefore, computing $55 - 48\frac{1}{2}$ will give the difference in height between the tallest and shortest player. Conclude that the difference in height between the tallest and shortest player is $6\frac{1}{2}$ inches.

- Ask students to use given information about an incomplete line plot to complete the line plot. For example, give students the prompt "A company that sells cans of vegetables measured the exact weights, in ounces, of seven cans of vegetables. The line plot represents the data collected, but one data point is missing.



If the total weight of all seven cans is $73\frac{3}{8}$ ounces, where should the missing point be plotted?" Discuss that determining where to plot the missing point is the same as finding the weight of the seventh can. To find the weight of the seventh can, add up the weights of the six cans displayed on the line plot and subtract this amount from the given total weight of $73\frac{3}{8}$ ounces. The line plot shows that the weights of the six given cans, in ounces, are $10\frac{2}{8}$, $10\frac{3}{8}$, $10\frac{4}{8}$, $10\frac{5}{8}$, $10\frac{5}{8}$, and $10\frac{5}{8}$. Using any strategy, students will compute $10\frac{2}{8} + 10\frac{3}{8} + 10\frac{4}{8} + 10\frac{5}{8} + 10\frac{5}{8} + 10\frac{5}{8} = 63$ ounces. Therefore, the weight of the seventh can is found by computing $73\frac{3}{8} - 63 = 10\frac{3}{8}$ ounces, indicating that the missing point should be plotted above $10\frac{3}{8}$ on the line plot.

Key Academic Terms:

line plot, measurement, fraction, mixed number, data set, visual model, maximum, minimum, scale, title, label, total, different, compute

Additional Resources:

- Video: [Fractions on a line plot]
- Article: [Line plot activities and resources]
- Activity: [Aluminum foil boats]

21a

Measurement
Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
21. Select and use an appropriate unit of measurement for a given attribute (length, mass, liquid volume, time) within one system of units: metric - km, m, cm; kg, g, l, ml; customary - lb, oz; time - hr, min, sec. a. Within one system of units, express measurements of a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

Guiding Questions with Connections to Mathematical Practices:

How can the relative size of measurement units within a system of units be compared?

M.P.7. Look for and make use of structure. Identify personal benchmarks to learn and practice relative sizes. For example, a paperclip has a mass of approximately 1 gram, which can be used as the benchmark to estimate the mass of other objects. Additionally, personal benchmarks can be found for a meter, a pound, an ounce, and a liter to help make sense of the measurements.

- Ask students to find personal benchmarks for common units of measurement using items from the classroom. Record the benchmark items in a table for reference.

Measurement	Benchmark
1 meter	height of desk
1 gram	mass of paperclip
1 pound	weight of a pair of shoes
1 liter	amount of water in a reusable water bottle
1 foot	length of textbook

- Ask students to compare the relative sizes of units within the same measurement system. For example, find how many ounces are in a pound by using a kitchen scale and piling one-ounce objects on it until the scale reads 1 pound. The same method will show how many grams are in a kilogram. To find the number of inches in a foot, have students measure an object that is one foot in length with an inch-ruler. The same method is used to find the number of feet in a yard or the number of centimeters or millimeters in a meter.

- Ask students to convert measurements of a larger unit in terms of a smaller unit within the US Customary system. For example, the measurement of 200 pounds in terms of ounces can be found by multiplying by the number of ounces in one pound, 16, by 200: $200 \times 16 = 3{,}200$ ounces. As an additional example, to find the number of seconds in 3 hours, first find the number of minutes in 3 hours: $3 \times 60 = 180$ minutes. Then, find the number of seconds in 180 minutes: $180 \times 60 = 10{,}800$ seconds.

M.P.8. Look for and express regularity in repeated reasoning. Notice patterns in metric prefixes that help relate one quantity to another and connect the metric system to place value. For example, "kilo" means one thousand, so a kilometer is 1,000 meters and a kilogram is 1,000 grams. Additionally, the same prefixes can be applied to other measurements in the metric system (e.g., liters and grams). Further, "cent" means $\frac{1}{100}$ and "milli" means $\frac{1}{1{,}000}$, so a centimeter is one hundredth of a meter and a millimeter is one thousandth of a meter.

- Ask students to use the pattern of metric units and place value understanding to write metric conversions. For example, the measurement of 4 kilograms can be written as 4,000 grams, since "kilo" means one thousand. And since "milli" means one thousandth, 5 liters can be written as 5,000 milliliters.

- Ask students to convert a measurement of a larger unit in terms of a smaller unit within the metric system. For example, find the number of milligrams in 76 kilograms. First find the number of grams by multiplying 76 by 1,000.

  $$76 \times 1{,}000 = 76{,}000 \text{ grams}$$

  Then 76,000 multiplied by 1,000 to find the number of milligrams.

  $$76{,}000 \times 1{,}000 = 76{,}000{,}000 \text{ milligrams}$$

  There are 76,000,000 milligrams in 76 kilograms.

How can tables be used to record relative measurements?

M.P.4. Model with mathematics. Record examples of equivalent measurements in a two-column table. For example, for a table with centimeters and meters, use the rule "divide by 100" when converting centimeters to meters and "multiply by 100" when converting meters to centimeters. Additionally, a two-column table can be used to show the conversion between US Customary units of measurement (e.g., pounds to ounces or feet to inches). Further, tables can include more than two columns to compare multiple units of measurement within the same system, like hours to minutes to seconds.

- Ask students to use a table to show multiple conversions of a measurement. For example, create a conversion table that shows the relationship between pounds and ounces. Notice the pattern that as the number of pounds increases by 1, the number of ounces increases by 16; therefore, there are 16 ounces in each pound.

Pounds	Ounces
1	16
2	32
3	48
4	64
5	80

- Ask students to use a table to show conversions of units of time. For example, create a table that shows the relationship between hours, minutes, and seconds. Notice the pattern that as the number of hours increases by 1, the number of minutes increases by 60, and the number of seconds increases by 3,600. Therefore, there are 60 minutes in an hour, and there are 3,600 seconds in an hour.

Hours	Minutes	Seconds
1	60	3,600
2	120	7,200
3	180	10,800
4	240	14,400
5	300	18,000

Key Academic Terms:

personal benchmark, relative size, prefix, equivalent measurement, conversion table, estimate, mass, metric system, comparisons, kilometer, meter, centimeter, gram, pound, ounce, liter, milliliter, hour, minute, second, US Customary system

Additional Resources:

- Lesson: Create conversion tables
- Lesson: Express metric mass measurements
- Lesson: Express metric capacity measurements
- Lesson: Solve multiplicative comparison word problems
- Activity: Measurement hunt

22a

Measurement
Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
22. Use the four operations to solve measurement word problems with distance, intervals of time, liquid volume, mass of objects, and money. a. Solve measurement problems involving simple fractions or decimals.

Guiding Questions with Connections to Mathematical Practices:

How can measurement word problems be represented and solved?

M.P.1. Make sense of problems and persevere in solving them. Write equations to solve measurement problems. For example, the problem "A recipe for one loaf of bread uses $\frac{1}{4}$ cup of sugar. How many cups of sugar are used to make 3 loaves of bread?" can be solved by writing the equation $\frac{1}{4} \times 3 = c$, and $c = \frac{3}{4}$ cup of sugar. Additionally, know that key words in a word problem indicate the operation that can be used to solve the problem. Further, use strategies such as decomposition of numbers to perform the operations indicated in a word problem.

- Ask students to solve a word problem by determining the operation the problem indicates and using a strategy to compute the operation indicated. For example, give students the prompt "The price of 4 lunches at Angie's Cafe is $38, where each lunch costs the same amount. What is the price for 1 lunch at Angie's Cafe?" Determine that since the problem involves splitting a total quantity into equal groups (or knowing the total cost for 4 equally priced lunches and wanting to know the cost for 1 lunch), the indicated operation is division. Thus, dividing the total cost for 4 lunches by 4 will give the cost for 1 lunch. Calculate $38 \div 4$ by expressing 38 as $36 + 2$ because 36 is the largest number less than 38 that is evenly divisible by 4. Since $36 \div 4$ is 9, there are $2 remaining to be divided into 4 equal parts, and so 0.50 is added to the $9. The cost for 1 lunch is therefore $9.50.

- Ask students to solve a measurement problem involving simple fractions. For example, give students the figure shown with this prompt: "A scale with clay blocks on each side is out of balance. Each clay block has a mass of 1 kilogram. How much mass needs to be moved from the right side of the scale to the left side of the scale to make it balanced?"

  Students can add to find the total mass of the two sides together, 9 kilograms, and then divide 9 by 2 to find that each side needs $4\frac{1}{2}$ kilograms for the scale to be balanced. Since the left side of the scale currently has 3 kilograms of mass, students can write the equation $3 + m = 4\frac{1}{2}$ where m represents the mass that needs to be added to the left side. Solving for m results in $m = 1\frac{1}{2}$; therefore, $1\frac{1}{2}$ kilograms need to be moved from the right side to the left side of the scale. Students can observe that removing $1\frac{1}{2}$ kilograms from the right side results in $4\frac{1}{2}$, and therefore the scale is balanced with $4\frac{1}{2}$ kilograms on each side. In this example, one of the clay blocks would have to be cut in half to make this possible.

Key Academic Terms:

operations, distance, intervals of time, liquid volume, mass, money, number line diagram, table, measurement scale, double number line, tape diagram, equation

Additional Resources:

- Lesson: Express metric capacity measurements
- Lesson: Solve multiplicative comparison word problems
- Lesson: Solve multistep measurement word problems
- Lesson: Solve multistep measurement word problems (continued)
- Activities: 4th grade measurement and data

22b

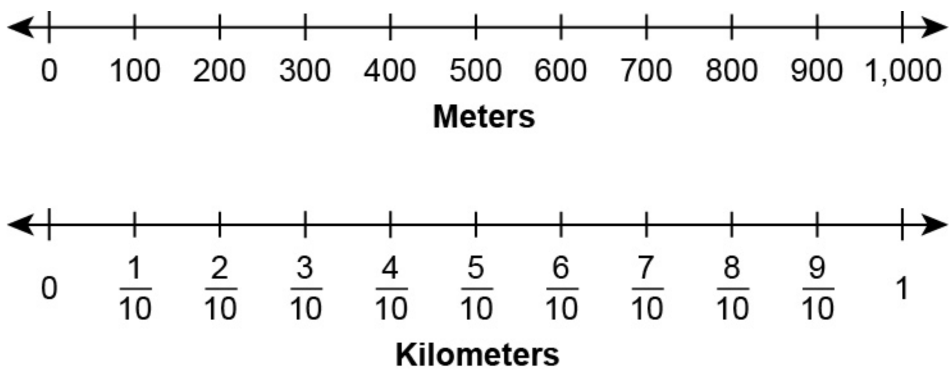
Measurement
Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
22. Use the four operations to solve measurement word problems with distance, intervals of time, liquid volume, mass of objects, and money. b. Solve measurement problems that require expressing measurements given in a larger unit in terms of a smaller unit.

Guiding Questions with Connections to Mathematical Practices:

How can measurements be expressed in units that are different sizes?

M.P.2. Reason abstractly and quantitatively. Express a measurement with two different units by using the conversion factor between the larger unit and the smaller unit. For example, in the metric system, conversion factors between larger units of measurement (e.g., meters) and smaller units of measurement (e.g., centimeters or millimeters) are powers of 10 (e.g., 100 or 1,000). Additionally, the US Customary units of measurement can be multiplied by conversion factors to find smaller units of measurement.

- Ask students to solve a word problem that contains different units in the metric system by expressing measurements given in a larger unit in terms of a smaller unit. For example, give students this prompt: "Lucas ran for $3\frac{1}{10}$ kilometers. Julie ran 250 meters less than Lucas. How many meters did Julie run?" Students determine that the units must be the same to answer the question. There are 1,000 meters in a kilometer, so Lucas' distance can be expressed in meters. The whole number of kilometers, 3, can be expressed as 3,000 meters, and $\frac{1}{10}$ of a kilometer can be expressed as 100 meters. This can be demonstrated using a double number line.

  Lucas' distance is therefore 3,100 meters. Julie's distance is described in the prompt as 250 meters less than the distance Lucas ran, and $3{,}100 - 250 = 2{,}850$. Julie's distance is therefore 2,850 meters.

- Ask students to solve a word problem that contains different units in the US Customary system by expressing measurements given in a larger unit in terms of a smaller unit. For example, give students this prompt: "Marla measures the length of her shadow twice: once during recess and again at the end of the school day. At recess, her shadow is 3 feet long. At the end of the day, her shadow is 4 feet and 2 inches long. How much longer, in inches, is her shadow at the end of the day?" To change the measurements into inches only, students can think of 1 foot as one group of size 12 inches. Therefore, the recess-shadow of 3 feet is the same as 3 groups of size 12 inches, which is 36 inches. The end-of-the-day-shadow is 4 feet 2 inches where the 4 feet can be thought of as 4 groups of size 12 inches, which is 48 inches. The 2 inches is added for a total of 50 inches. The difference can be found by subtracting 50 – 36 for an answer of 14 inches.

Key Academic Terms:

operations, distance, intervals of time, liquid volume, mass, money, number line diagram, table, measurement scale, double number line, tape diagram

Additional Resources:

- Lesson: Express metric capacity measurements
- Lesson: Solve multiplicative comparison word problems
- Lesson: Solve multistep measurement word problems
- Lesson: Solve multistep measurement word problems (continued)
- Activities: 4th grade measurement and data

22c

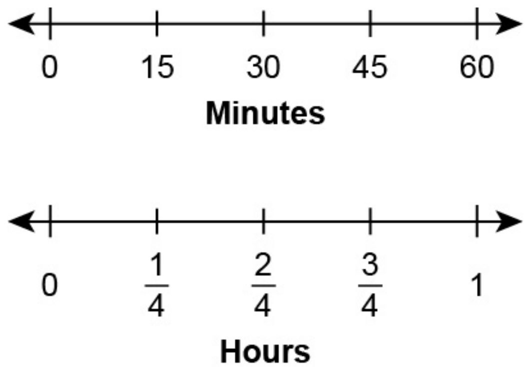
Measurement
Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
22. Use the four operations to solve measurement word problems with distance, intervals of time, liquid volume, mass of objects, and money. c. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Guiding Questions with Connections to Mathematical Practices:

How can measurement word problems be represented and solved?

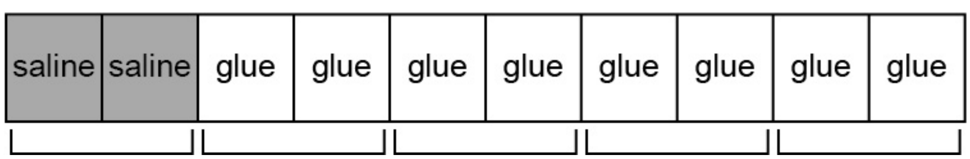
M.P.4. Model with mathematics. Draw and use diagrams to solve measurement problems. For example, the problem "Patrick walks $\frac{4}{10}$ of a kilometer. Angela walks 3 times farther than Patrick walks. How many more meters does Angela walk than Patrick walks?" can be solved by using a double number line to show the connection between kilometers and meters and show how to use multiplication and subtraction to solve the problem. Additionally, know that double number lines and tape diagrams are strategies that can be used when working with measurements of different units.

- Ask students to solve a measurement problem that contains units which measure the same quantity using a diagram. For example, give students the prompt "On Friday, Steph read her book for $\frac{3}{4}$ hour. On Saturday, Steph read for 3 times as long as she read on Friday. How many minutes, in total, did Steph read between Friday and Saturday?" Determine that $\frac{3}{4}$ hour can be represented as 45 minutes using a double number line.



Calculate the number of minutes Steph practiced on Saturday by multiplying 45 by 3 to get 135 minutes. Determine that Steph read a total of 180 minutes between Friday and Saturday by adding 45 and 135.

- Ask students to solve a measurement problem using a tape diagram. For example, give students the prompt "Elaine put saline solution and glue in a bowl to make slime. The mixture made 10 ounces of slime in all. She put 4 times as much glue as saline in the bowl. How many ounces of glue did she put in the bowl?" The tape diagram is in 10 partitions to represent 10 ounces of slime.



Determine that there will be 2 ounces of saline solution and 8 ounces of glue in the slime by comparing the relative size of each portion of the tape diagram.

Key Academic Terms:

operations, distance, intervals of time, liquid volume, mass, money, number line diagram, table, measurement scale, double number line, tape diagram

Additional Resources:

- Lesson: Express metric capacity measurements
- Lesson: Solve multiplicative comparison word problems
- Lesson: Solve multistep measurement word problems
- Lesson: Solve multistep measurement word problems (continued)
- Activities: 4th grade measurement and data

23

Measurement
Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
23. Apply area and perimeter formulas for rectangles in real-world and mathematical situations.

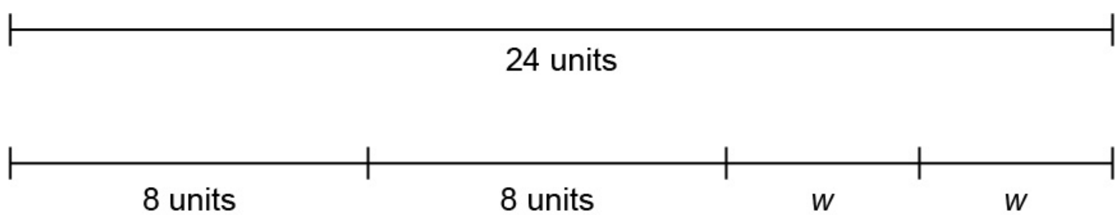
Guiding Questions with Connections to Mathematical Practices:

How can visual models be used to apply the area and perimeter formulas?

M.P.4. Model with mathematics. Draw visual representations of rectangles to apply the area and perimeter formulas. For example, the problem "A rectangle-shaped playground has an area of 56 square meters and a width of 7 meters. What is the length of the playground?" can be solved by drawing a sketch of the rectangle to determine that the length and width are multiplied when finding area, so $56 = L \times 7$; therefore, $L = 8$ meters. Additionally, know that the perimeter of a rectangle is the linear distance around the rectangle, while the area of a rectangle measures the space inside the rectangle, and that this can be demonstrated using visual models.

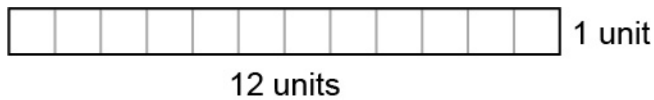
- Ask students to use a visual representation of a rectangle to determine an unknown measurement of the rectangle. For example, give students the prompt "The rectangle in the diagram has a perimeter of 24 units. What is the width of the rectangle?"

Explain that the sides of the rectangle can be shifted so that they form a straight line. This line must be 24 units long because the perimeter is 24 units, and the perimeter measures the distance around the shape.

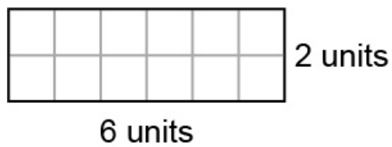


The sum of the two sides representing the width of the rectangle must be 8 units because the sum of all the sides must be 24 units, and $8 + 8 + 8 = 24$. This means that each width of the rectangle must be 4 units, because there are two equal widths with a sum of 8. The width of the rectangle is therefore 4 units.

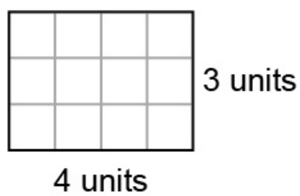
- Ask students to use visual representations of rectangles to solve a problem involving rectangles. For example, give students the question "How many unique rectangles have an area of 12 square units and side measurements that are whole numbers?" Determine that the area of a rectangle measures the space inside the rectangle by multiplying the length by the width. Start with a visual representation of the rectangle using side lengths of 1 unit and 12 units. Since the product of 1 and 12 is 12, this rectangle must have an area of 12 square units.



Explain that a rectangle with an area of 12 square units could also have sides measuring 2 units and 6 units and demonstrate this using a visual representation.



Determine that a final rectangle with an area of 12 square units could be constructed with side lengths of 3 units and 4 units.



Conclude that there are 3 unique rectangles with an area of 12 square units that have whole number side measurements.

How can area and perimeter formulas be used to create and solve equations with an unknown?

M.P.7. Look for and make use of structure. Represent area and perimeter problems as equations with one unknown. For example, the problem "A rectangle has a perimeter of 14 units and a length of 3 units. What is the width of the rectangle?" can be solved by writing and solving the equation that represents the problem, $14 = 3 + 3 + w + w$, by recognizing that perimeter is the addition of the four sides of a rectangle. Additionally, observe that the formulas $P = 2(l + w)$ and $P = 2l + 2w$ both represent the relationship between the perimeter (P), length (l), and width (w), of a rectangle. Further, the formula $A = l \times w$ represents the relationship between the area (A), length (l), and width (w) of a rectangle.

- Ask students to determine an unknown quantity in the perimeter formula of a rectangle given the value of the other two quantities. For example, give students the prompt "The perimeter of a rectangle is 26 units. The width of the rectangle is 4 units. What is the length of the rectangle?" Determine that the perimeter of a rectangle is the sum of its four sides, and there are two unique values, the length and the width, that make up these four sides. Write the equation $26 = 4 + 4 + l + l$ to represent the situation, where l is the length of the rectangle. The right side of the equation is the sum of 8 and $l + l$, and it is known that the sum must be 26 based on the equation. Explain that $l + l$ must therefore be equal to 18 because $26 = 8 + 18$. As 18 represents the total of two lengths, explain that it can be divided by 2 to determine the measure of a single length of the rectangle. The length of the rectangle is, therefore, 9 units.

- Ask students to determine an unknown quantity in the area formula of a rectangle given the value of the other two quantities. For example, give students the prompt "The area of a rectangle is 48 square units, and the length of the rectangle is 8 units. What is the width of the rectangle?" Determine that the area of a rectangle can be represented by the product of the length and width of the rectangle. Write the equation $48 = 8 \times w$ to represent the situation where w is the width of the rectangle. In the equation, the area that was given is equivalent to an expression that also represents the area: the length multiplied by the width. The width of the rectangle is represented using a letter as a variable, as it is the unknown value that is being asked for. Explain that the width of the rectangle must be a number such that when that number is multiplied by 8, the product is 48. This value can be determined using reasoning or by determining that 48 can be divided by 8. Conclude that the unknown value in the equation is 6, so the width of the rectangle is 6 units.

Key Academic Terms:

area, perimeter, formula, rectangle, length, width, equation, square units, dimension, variable, distance

Additional Resources:

- Activities: Perimeter and Area Song and activities for kids
- Article: Perimeter and area of rectangles

24

Measurement
Geometric measurement: understand concepts of angle and measure angles.
24. Identify an angle as a geometric shape formed wherever two rays share a common endpoint.

Guiding Questions with Connections to Mathematical Practices:

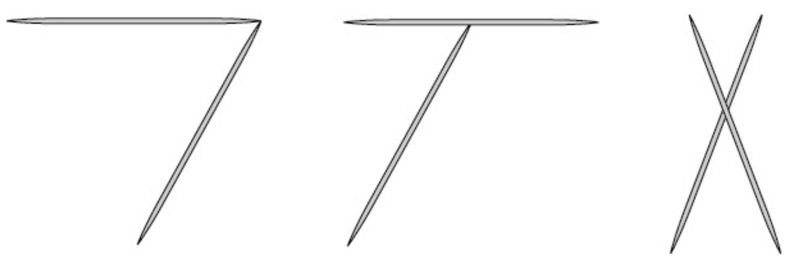
How is an angle defined?

M.P.6. Attend to precision. Know that an angle is formed by two rays that share a common endpoint. For example, a vertex of a parallelogram is part of an angle because the two adjacent sides are parts of rays and the vertex is the common endpoint shared by the two rays. Additionally, angles can be formed by lines and line segments.

- Provide students with a complete or partial list of the Roman alphabet. Ask them to circle all the letters that contain 1 or more angles. For example, in the list shown, students would circle the letters A, E, H, V, and W.

A C E H J O S U V W

- Provide students with two or more toothpicks. Ask them to create figures that contain a particular number of angles. For example, if students have two toothpicks, then they could create figures with 1, 2, or 4 angles as shown.



Key Academic Terms:

angle, endpoint, ray, intersect, point, vertex, adjacent, line segment, parallelogram

Additional Resources:

- Lesson: Angle measure and length measurement
- Article: Hands-on measurement activities to engage students
- Video: Grade 4 math 11.2, degrees of an angle, circle
- Article: 5 activities for teaching angles

25

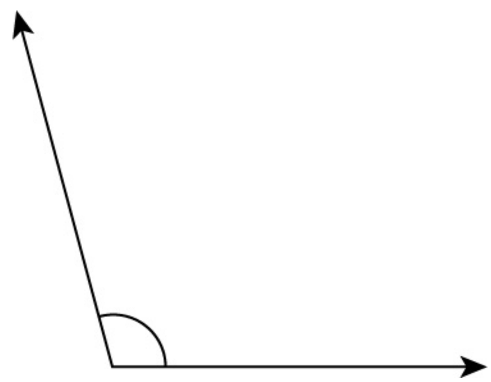
Measurement
Geometric measurement: understand concepts of angle and measure angles.
25. Use a protractor to measure angles in whole-number degrees and sketch angles of specified measure.

Guiding Questions with Connections to Mathematical Practices:

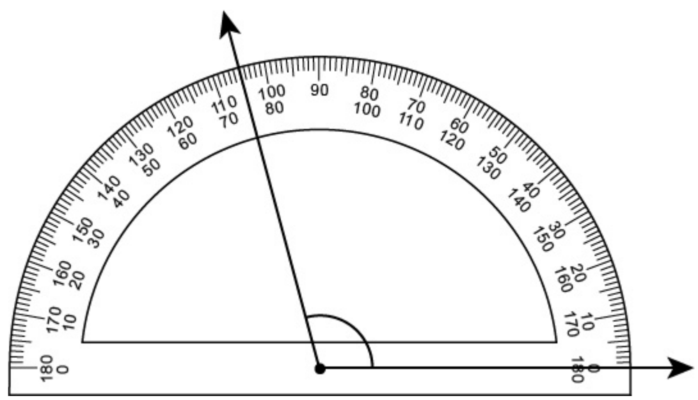
How are angles measured?

M.P.5. Use appropriate tools strategically. Demonstrate how to use a protractor to measure angles in different orientations to the nearest degree. For example, align the vertex of the angle with the correct point on the protractor, align one leg of the angle with the 0° mark on the protractor, and read where the other leg is located on the protractor. Additionally, observe that when one of the rays that make up an angle is not aligned with the 0° mark on the protractor, the measure of the angle can be determined by finding the difference of the marks on the protractor at each ray.

- Ask students to measure an angle by aligning one of the rays with the 0° mark on the protractor. For example, give students the angle shown.

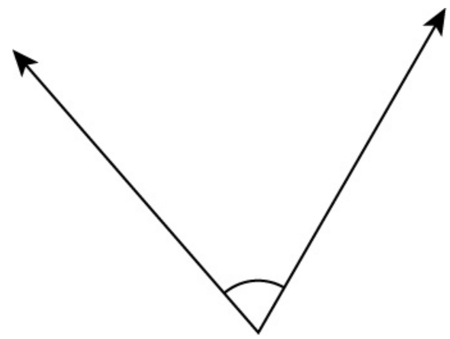


Align the 0° mark on a protractor with one of the rays to ensure that the other ray will intersect the protractor at its degree measurement.

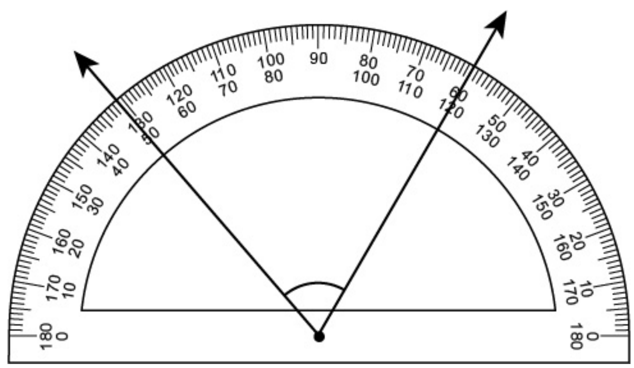


Read the degree measurement on the protractor and note that the protractor has two sets of values—the inner numbers and the outer numbers—and it is important to identify which set of values to use. In this case, the horizontal ray that aligns with the 0° mark points to the right, thus passing through the 0° mark of the outer numbers. Therefore, the outer numbers should be used to name the degree measure of the other ray. Since the other ray intersects the protractor exactly halfway between 100° and 110°, it can be concluded that the measure of this angle is 105°.

- Ask students to measure an angle by aligning one of the rays with any mark on the protractor and finding the difference between the marks at which the rays intersect the protractor. For example, give students the following angle.



Align one of the rays with a mark on the protractor. Using a mark that is a multiple of 10 is a useful strategy. Ensure that the other ray also intersects the measurement portion of the protractor. If it does not, consider extending the ray so it does intersect the measurement portion of the protractor.



Read the degree measurements for both rays, using the same set of values for both rays. In the diagram, the rays intersect the inner numbers at 120° and 49°, or they intersect the outer numbers at 131° and 60°. Therefore, the angle measures 120° – 49° = 71° or 131° – 60° = 71°.

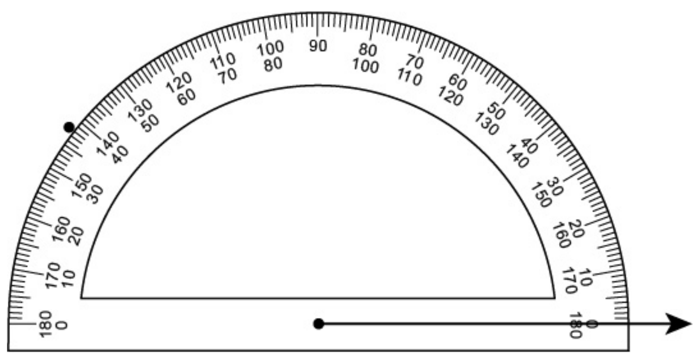
How can angles be drawn to a given measurement?

M.P.5. Use appropriate tools strategically. Draw an angle of a given size using a variety of tools. For example, draw a ray on a piece of paper and use a protractor to construct a 108° angle. Additionally, know that to construct an angle given one of the rays that make up the angle, use a protractor to measure the degrees in the angle and mark a point for the other ray, then remove the protractor and use the straight edge to draw the other ray.

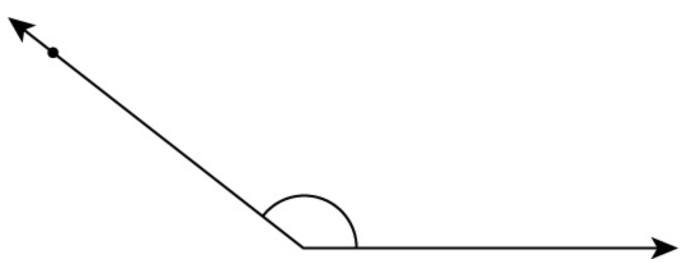
- Ask students to draw an angle of a given measurement using a protractor. For example, ask students to draw a 142° angle. Begin by drawing a ray using a straight edge.

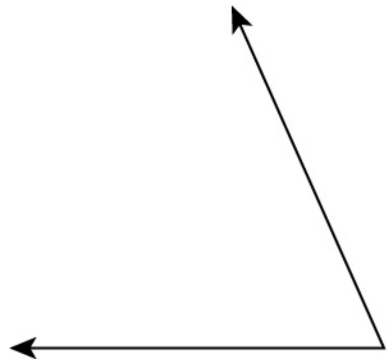
  Align a protractor with the ray such that the vertex of the ray is centered and the ray intersects the 0° mark. Since this ray intersects the 0° mark of the outer set of numbers, use the outer set of numbers to identify and mark a point at 142°.

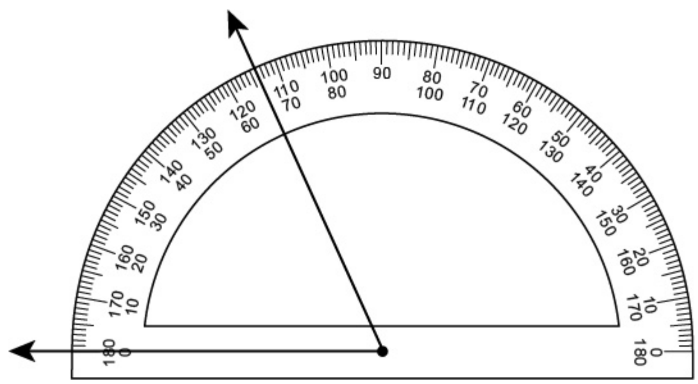
  Remove the protractor and use the straight edge to draw the ray that connects the endpoint of the first ray with the point drawn. Then, label the drawing with an arc to indicate the 142° angle.

- Ask students to identify errors made in the drawing of an angle and to correctly draw the angle. For example, give students the prompt "A student attempted to draw a 114° angle using a protractor. The student's angle is shown.



Identify the error the student made and explain how to correctly draw an angle measuring 114°." Instruct students to use a protractor to measure the incorrect angle.



Determine that the error the student made was using the incorrect set of values along the edge of the protractor. Since the ray that aligns with the 0° mark intersects the 0° mark of the inner set of values, the inner set of values must also be used to identify the placement of the other ray.

Key Academic Terms:

measure, angle, degree, protractor, align, leg, ray, endpoint, point, vertex, angle measure, arc, mark, intersect

Additional Resources:

- Lesson: [Angle measure and length measurement]
- Activity: [Measuring angles]

26a

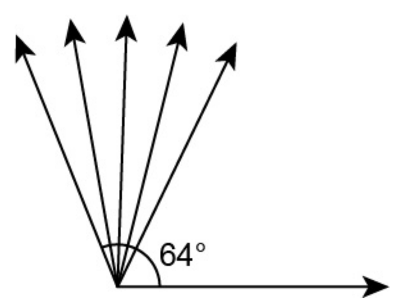
Measurement
Geometric measurement: understand concepts of angle and measure angles.
26. Decompose an angle into non-overlapping parts to demonstrate that the angle measure of the whole is the sum of the angle measures of the parts. a. Solve addition and subtraction problems on a diagram to find unknown angles in real-world or mathematical problems.

Guiding Questions with Connections to Mathematical Practices:

How are angle measures additive?

M.P.7. Look for and make use of structure. Explain that angles can be decomposed into smaller angles and that the sum of the measurements of those angles equals the measurement of the larger angle. For example, a 180-degree angle is composed of 180 one-degree angles, or $180° = 60° + 30° + 90°$, or any other combination of angles that add up to 180°. Additionally, observe that the smaller angles share common rays but do not have overlapping parts when composing a larger angle.

- Ask students to determine the measure of an angle composed of smaller angles. For example, give students the prompt "The smallest angles in the diagram each measure 12°. What is the measure of the largest angle in the diagram?"



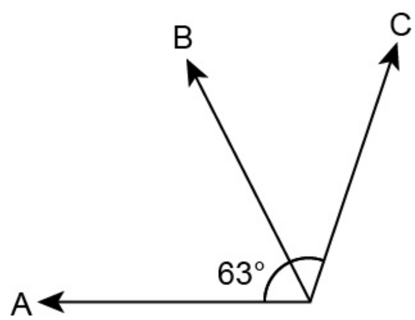
Determine that the largest angle is made up of four angles that measure 12° and one angle that measures 64°. Explain that the sum of the measures of these angles is equivalent to the measure of the largest angle. Add the measures of the angles that make up the largest angle to get $12° + 12° + 12° + 12° + 64° = 112°$. The largest angle therefore measures 112°.

- Ask students to determine ways in which an angle can be decomposed into smaller angles. For example, give students the prompt "An angle that measures 133° is decomposed into an angle that measures 58° and three other angles. Give two examples of possible measures of these three other angles." Determine that the sum of the three angles must be 75° because 133° – 58° = 75°. Explain that there are many ways to decompose the remaining 75° into three angles. One way to decompose 75° into three angles would be to use two angles that each measure 1° and one angle that measures 73° because 1° + 1° + 73° = 75°. Another way to decompose 75° into three angles would be to divide it evenly into three parts so that each of the three angles measures 25°.

How can the additive nature of angle measures be used to solve real-world and mathematical problems?

M.P.2. Reason abstractly and quantitatively. Calculate the measurement of an angle composed of other angles. Find a missing angle measure when given the whole measure of an angle and the measure of part of the angle for real-world and mathematical problems. For example, for a 90° angle that has been decomposed into three angles, two of which measure 40° and 28°, the measurement of the third angle can be found by solving 90° – 40° – 28°. Additionally, use context in a word problem to determine which operations or inverse operations can be used to solve the problem.

- Ask students to solve a real-world problem by composing or decomposing an angle. For example, give students the prompt "Three ships set out on straight paths. The rays in the diagram represent the path of each ship. The angle between the paths of ship C and ship A is 109°. What is the angle between the paths of ship C and ship B?"

  Determine that the angle between the paths of ship A and ship C is composed of the angle between the paths of ship A and ship B (63°), as well as the angle between the paths of ship B and ship C. Explain that to find the measure of the angle between the path of ship B and C, subtract the 63° angle from the larger 109° angle. This operation can be performed because the two smaller angles must add to 109°, so 109° – 63° will give the measure of the other smaller angle. The angle between the path of ship C and ship B is therefore 46°.

- Ask students to solve a problem by composing or decomposing an angle using reasoning and context in the problem. For example, give students the prompt "An angle that measures 80° is decomposed into four angles. The two smallest angles each measure 10°. The next largest angle measures half as many degrees as the largest angle. What is the measure of the largest angle?" Determine that the angles whose measures are not specified must add to 60° because 80° – 20° = 60°. Explain that because one of the angles is equal to half of the other, the 60° can be split evenly three ways so that one angle measures that value and the other measures twice that value. Divide 60° by 3 to get 20°. The largest of the angles that make up the 80° angle must therefore measure 40° because it must be twice as large as the 20° angle.

Key Academic Terms:

angle, additive, decompose, addition, subtraction, diagram, addend, sum, angle measure, equation, compose, nonoverlapping

Additional Resources:

- Activity: Race around the world: an angles game (protractor practice)
- Activity: Finding an unknown angle

27

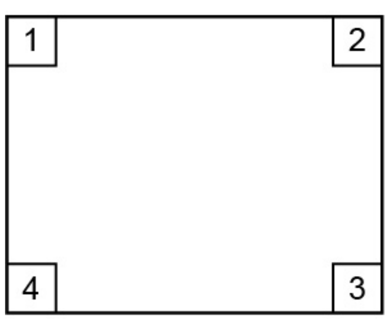
Geometry
Draw and identify lines and angles, and identify shapes by properties of their lines and angles.
27. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines, and identify these in two-dimensional figures.

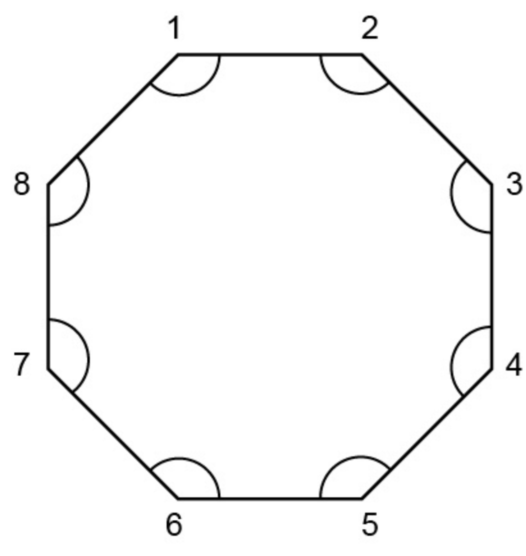
Guiding Questions with Connections to Mathematical Practices:

What are the characteristics of points, lines, line segments, rays, acute angles, right angles, obtuse angles, perpendicular lines, and parallel lines?

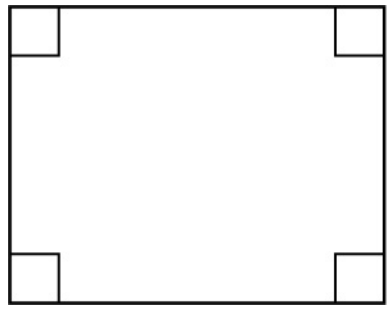
M.P.6. Attend to precision. Describe the characteristics of the given figures. For example, an obtuse angle is two rays that meet at a point called a vertex with an angle measure greater than 90 degrees. Additionally, a line is perpendicular to another line if the two lines meet at a 90° angle, and parallel lines are always the same distance apart and never intersect one another.

- Ask students to describe the angles in a given figure. For example, a rectangle has four right angles because there are four vertices at which line segments meet at a 90° angle. An octagon, like a stop sign, creates eight obtuse angles because there are eight vertices at which line segments meet to form angles greater than 90°.





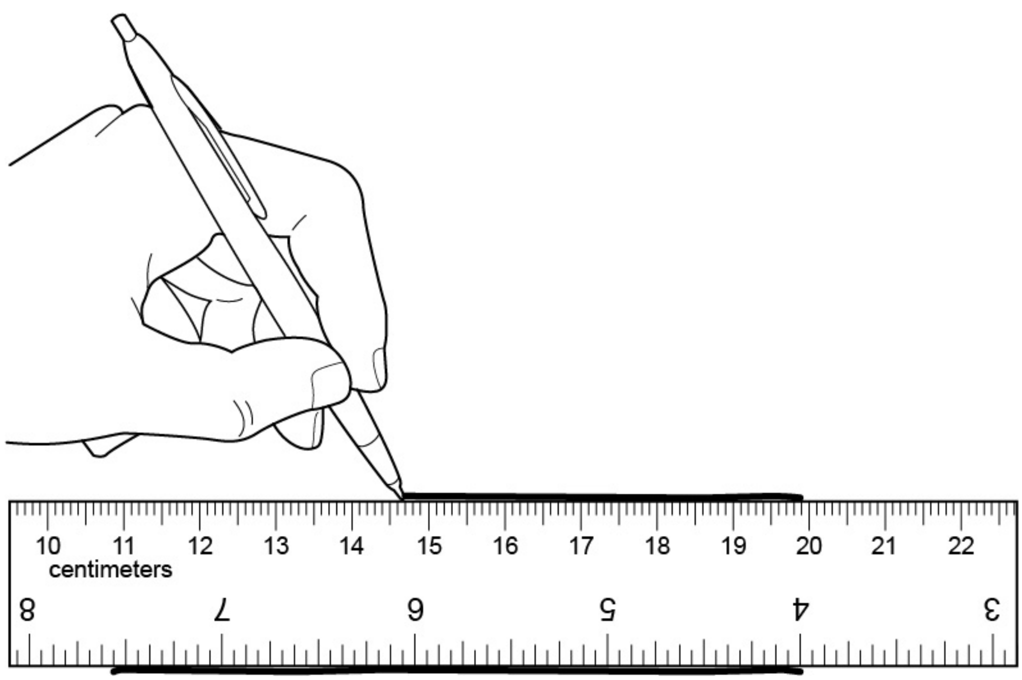
- Ask students to describe the line segments used to create a figure. For example, a rectangle consists of two sets of parallel line segments because each set of line segments is the same distance apart in the figure, and a rectangle consists of four pairs of perpendicular line segments because there are four vertices at which line segments meet at 90° angles.



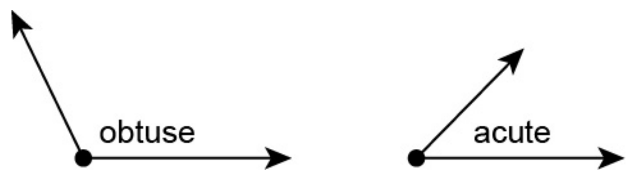
How can points, lines, line segments, rays, acute angles, right angles, obtuse angles, perpendicular lines, and parallel lines be drawn?

M.P.5. Use appropriate tools strategically. Draw a given figure correctly using a variety of tools. For example, use a ruler, paper, and pencil to draw two points and connect them to make a line segment. Additionally, use arrows at the ends of drawn line segments to create rays and lines.

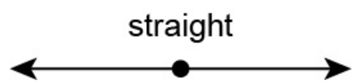
- Ask students to use a pencil, paper, and ruler to draw a set of parallel lines. For example, use a pencil to trace along each side of a ruler to create two parallel lines because the distance between the lines will always be the same (exactly the width of the ruler).



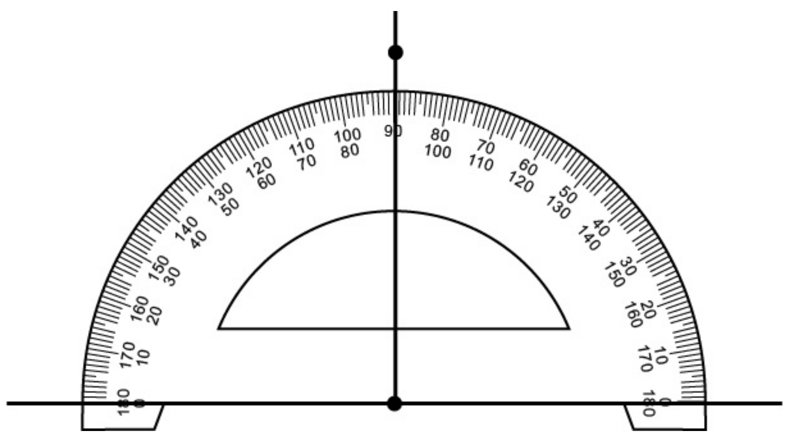
- Ask students to use a pencil, paper, and ruler to draw combinations of rays that create angles. For example, a plotted point with rays extending in different directions can create an acute angle, an obtuse angle, or a straight angle, depending on the measure of the angle created between the rays.



A straight angle has rays pointing in opposite directions.



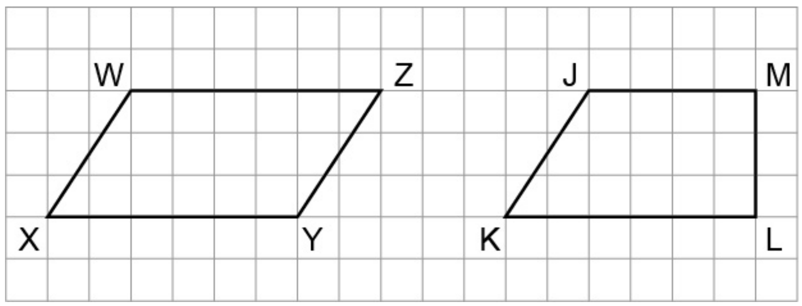
- Ask students to use a protractor to draw right angles and perpendicular lines. For example, draw a line along the straight edge of a protractor and then draw a point in the center of the straight line and a point at the 90° mark. Connect the two points to create two 90° (right) angles.



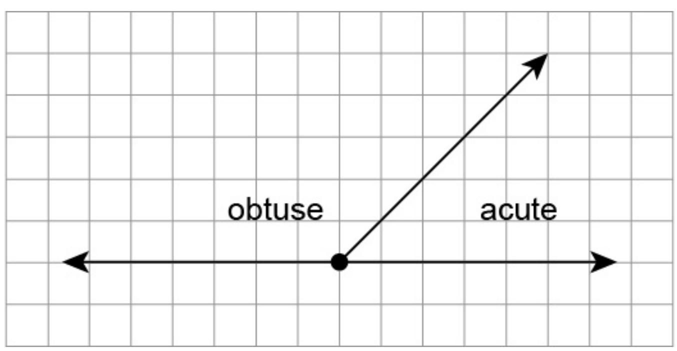
How can points, lines, line segments, rays, acute angles, right angles, obtuse angles, perpendicular lines, and parallel lines be distinguished in two-dimensional shapes?

M.P.6. Attend to precision. Identify the given figures in two-dimensional shapes. For example, in rectangle ABCD, identify that angle ABC is a right angle and that lines AB and CD are parallel. Additionally, in right triangle PQR, angle PQR is a right angle and angles QRP and RPQ are both acute.

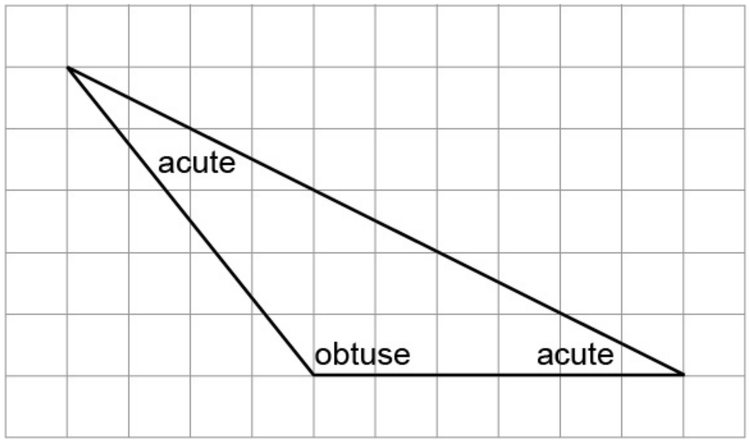
- Ask students to identify sets of parallel line segments in quadrilaterals. For example, in parallelogram WXYZ there are two sets of parallel line segments (WX and YZ, as well as WZ and XY), and in trapezoid JKLM there is one set of parallel line segments (KL and JM).



- Ask students to identify acute angles and obtuse angles in a figure. For example, given a line and ray that create non-right supplementary angles, students should identify which is acute and which is obtuse.



- Ask students to describe the angles and line segments in a given triangle. For the example shown in the image below, the triangle consists of three line segments with no pairs of parallel or perpendicular line segments, as well as two acute angles and one obtuse angle.



Key Academic Terms:

point, line, line segment, ray, angle, right, acute, obtuse, straight angle, perpendicular, parallel, two-dimensional, characteristics, vertex, naming conventions (i.e., rectangle ABCD or triangle ABC), arc, mark

Additional Resources:

- Video: [Angles song]
- Video: [Types of triangles]
- Video: [Camp Quadrilaterals]

28a

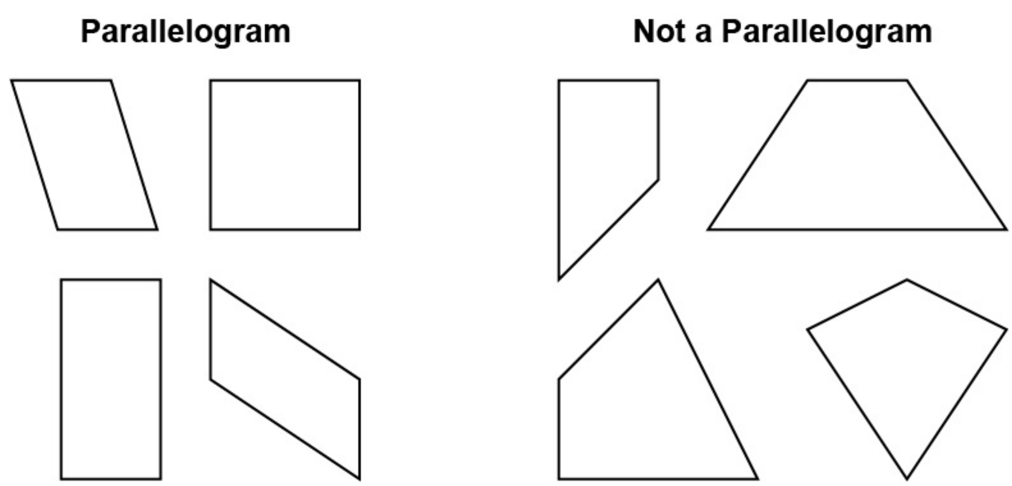
Geometry
Draw and identify lines and angles, and identify shapes by properties of their lines and angles.
28. Identify two-dimensional figures based on the presence or absence of parallel or perpendicular lines or the presence or absence of angles of a specified size. a. Describe right triangles as a category, and identify right triangles.

Guiding Questions with Connections to Mathematical Practices:

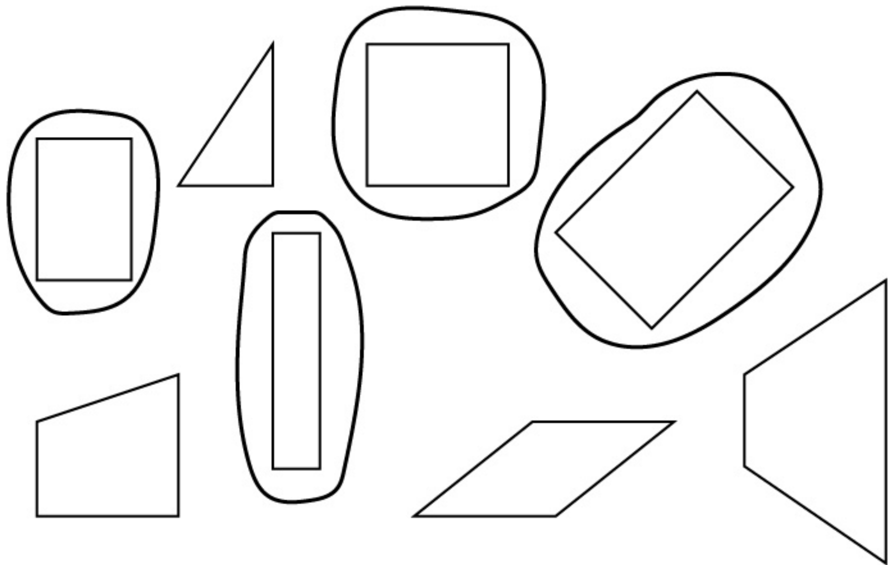
How can angle sizes, parallel lines, and/or perpendicular lines be used to categorize two-dimensional shapes?

M.P.7. Look for and make use of structure. Sort two-dimensional figures based on angle sizes or presence of parallel and/or perpendicular lines. For example, given a group of regular polygons, sort shapes into categories based on angle size as well as presence of parallel lines. Additionally, given a group of triangles, sort figures into categories based on the absence or presence of a right angle.

- Ask students to sort a set of quadrilaterals into “parallelogram” or “not a parallelogram” based on the number of sets of parallel lines in the figure. In the example shown, eight figures have been grouped based on whether they have two sets of parallel sides.

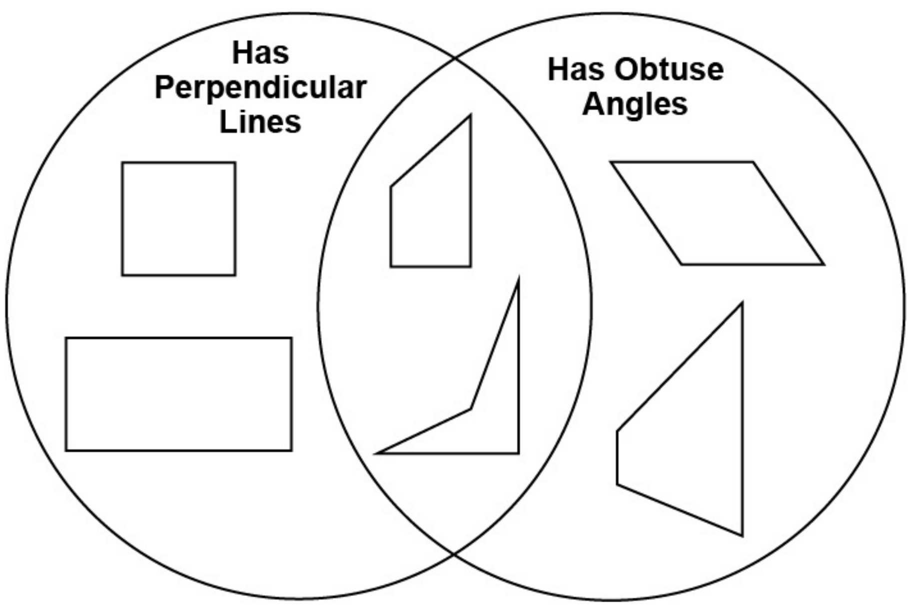


- Ask students to identify rectangles in a set based on the presence of four right angles. In the example shown, the rectangles have been circled by the student.



M.P.6. Attend to precision. Classify and name shapes using more than one characteristic. For example, a four-sided shape with opposite sides parallel and four right angles is both a parallelogram and a rectangle. Additionally, a triangle with three sides of equal length is equilateral and also acute because all three angles are less than 90°, specifically 60°.

- Ask students to sort quadrilaterals based on two characteristics. For example, give students a Venn diagram and ask them to create examples of shapes in each region. Some possible shapes are included.

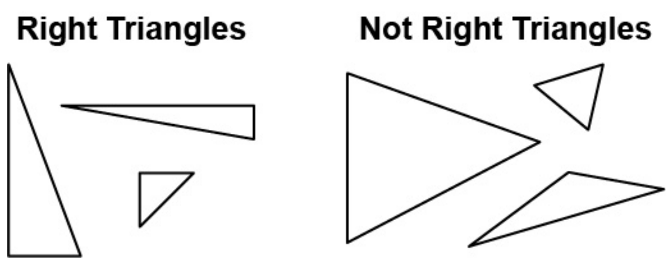


- Ask students to classify triangles based on perpendicular lines and angle measures. For example, ask students to sort triangles based on whether they have perpendicular sides and based on the number of equal angles they have. Know that a triangle with perpendicular sides and two equal angles is a right isosceles triangle and a triangle with perpendicular sides and no equal angles is a right scalene triangle.

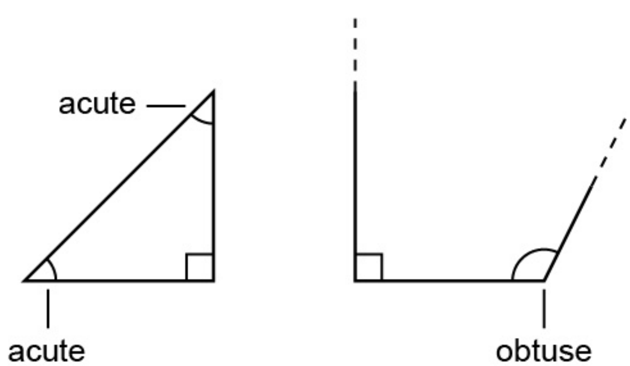
What are the characteristics of a right triangle?

M.P.7. Look for and make use of structure. Identify right triangles. Describe a triangle with one right angle and two acute angles as a right triangle. For example, given a group of triangles, identify which triangles are right triangles by looking for the triangles that have one right angle. Additionally, observe that a right triangle cannot have an obtuse angle.

- Ask students to sort a set of triangles into "right" or "not right" based on the presence or absence of a right angle. In the example shown, six figures have been grouped based on whether they have a right angle.



- Ask students to construct right triangles with certain characteristics and observe that there are some characteristics that a right triangle cannot have. For example, using graph paper and a pencil, an isosceles right triangle can be constructed and described as having one right angle and two acute angles, but an obtuse right triangle cannot be constructed because a triangle cannot have a right angle and an obtuse angle.



Key Academic Terms:

classify, categorize, characteristics, two-dimensional, polygon, angle, acute, obtuse, right angle, scalene, isosceles, equilateral, equiangular, parallel, perpendicular, regular polygon, parallelogram, rhombus, right angle symbol

Additional Resources:

- Activity: Geometry vocab sort
- Video: Types of triangles
- Video: Camp Quadrilaterals

29a

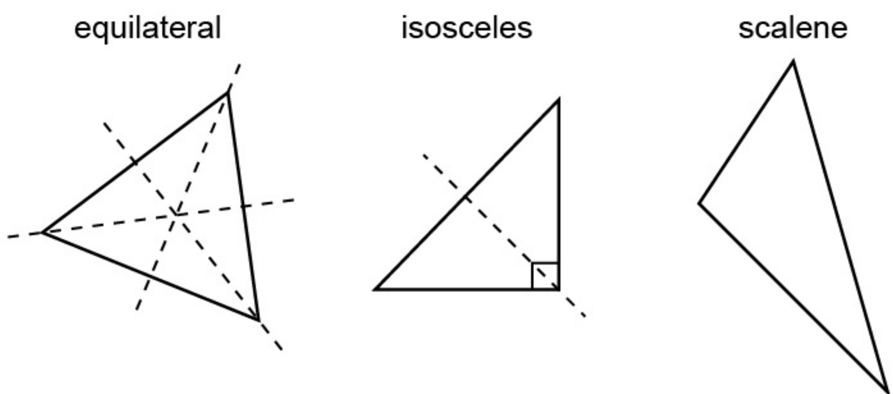
Geometry
Draw and identify lines and angles, and identify shapes by properties of their lines and angles.
29. Define a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. a. Identify line-symmetric figures and draw lines of symmetry.

Guiding Questions with Connections to Mathematical Practices:

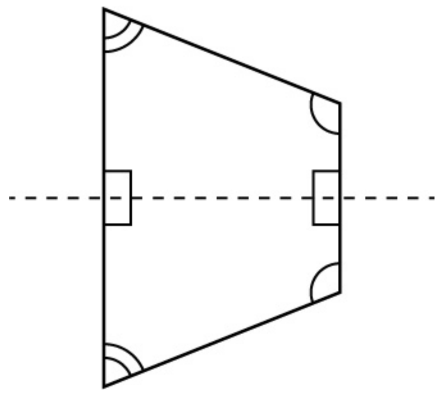
How can a line of symmetry be identified in a figure?

M.P.6. Attend to precision. Define, identify, and draw lines of symmetry by folding a shape. For example, fold a square in half four different ways to draw its four lines of symmetry. Also, for any shape, fold the paper in half to see if both halves are equal or are mirror images. Additionally, some figures have one line of symmetry (like an isosceles triangle), multiple lines of symmetry (like a rectangle) or an infinite number of lines of symmetry (like a circle).

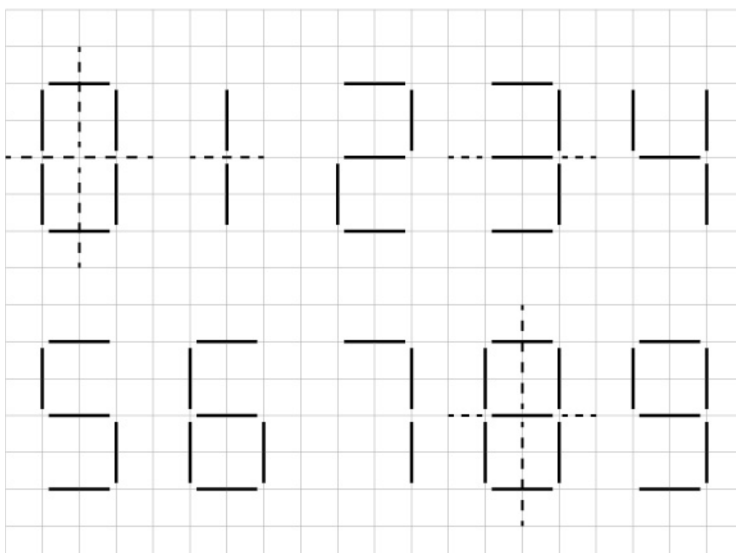
- Ask students to fold paper figures in half to determine lines of symmetry. For example, given three triangles (one equilateral, one isosceles, and one scalene), observe that the equilateral triangle has three lines of symmetry, the isosceles has one, and the scalene has none.



- Ask students to identify characteristics of the matching parts when a line of symmetry is drawn on a figure. For example, when a line of symmetry is drawn on an isosceles trapezoid, each half has two right angles, one acute angle, and one obtuse angle.

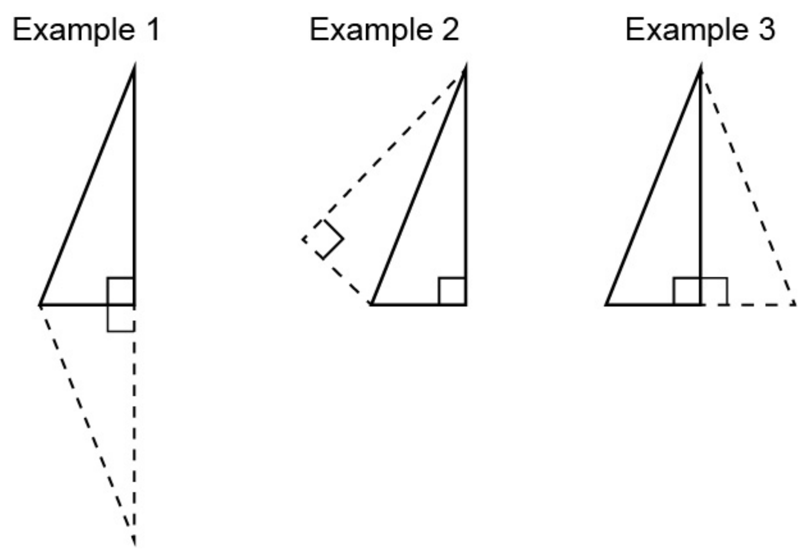


- Ask students to identify which figures in a set have a line of symmetry. For example, of the set of digital display numerals from zero to nine, the zero, one, three, and eight each have at least one line of symmetry.

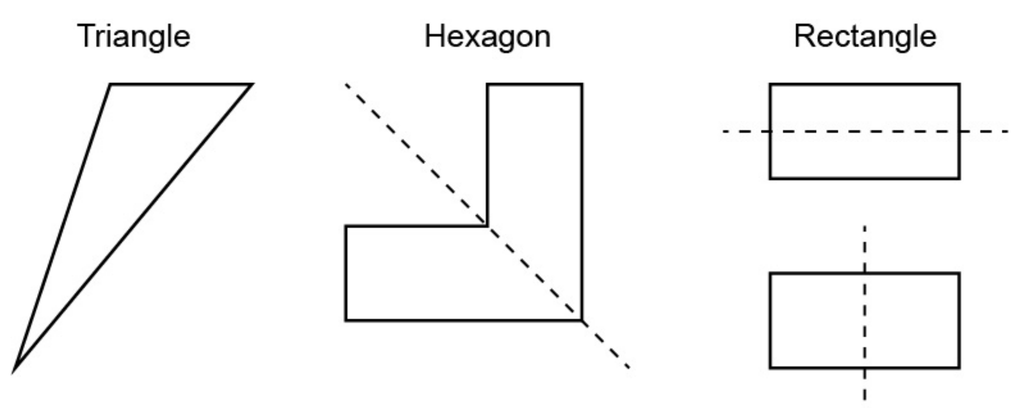


M.P.4. Model with mathematics. Given half of a two-dimensional figure on grid paper, draw the other half so that the two sides are symmetrical. Draw the line of symmetry that separates the two halves. For example, given half of a picture of a four-leaf clover, draw the other half of the picture. Additionally, given a two-dimensional figure, draw a line of symmetry such that each side is a mirror image of the other, and know there may be more than one line of symmetry.

- Ask students to use the edge of a figure as a line of symmetry and to complete the other half of the new symmetrical figure. For example, given a right scalene triangle, three different lines of symmetry can be drawn, resulting in the creation of three different symmetrical images.



- Ask students to draw a line of symmetry, if possible, for a figure, and to determine if more than one line of symmetry exists for the figure. In the example shown, the figure on the left has no line of symmetry, the figure in the middle has one line of symmetry drawn, and the figure on the right (shown twice) has two lines of symmetry.



Key Academic Terms:

symmetry, line of symmetry, two-dimensional figure, equal halves, mirror image, arc, mark, angle measure

Additional Resources:

- Video: Symmetry Land
- Worksheets: Symmetry worksheets
- Article: Hands-on geometry
- Lesson: Exploring line symmetry
- Book: Leedy, L. (2013). *Seeing symmetry*. New York, NY: Holiday House.

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February 15, 2022