ACAP

Alabama Comprehensive Assessment Program

SUMMATIVE

Grade 5
Mathematics

Alabama Educator Instructional Supports

Alabama Course of Study Standards

DATA RECOGNITION DRC CORPORATION

February 15, 2022

Table of Contents

Introduction

The *Alabama Instructional Supports: Mathematics* is a companion to the 2019 *Alabama Course of Study: Mathematics* for Grades K–12. Instructional supports are foundational tools that educators may use to help students become independent learners as they build toward mastery of the *Alabama Course of Study* content standards. **Instructional supports are designed to help educators engage their students in exploring, explaining, and expanding their understanding of the content standards.**

The content standards contained within the course of study may be accessed on the Alabama State Department of Education (ALSDE) website: https://www.alabamaachieves.org/. When examining these instructional supports, educators are reminded that content standards indicate minimum content—what all students should know and be able to do by the end of each grade level or course. Local school systems may have additional instructional or achievement expectations and may provide instructional guidelines that address content sequence, review, and remediation.

The instructional supports are organized by standard. Each standard's instructional support includes a statement of the content standard, guiding questions with connections to mathematical practices, key academic terms, and additional resources.

Content Standards

The content standards are the statements from the 2019 *Alabama Course of Study: Mathematics* that define what all students should know and be able to do at the conclusion of a given grade level or course. Content standards contain minimum required content and complete the phrase "Students will ______."

Guiding Questions with Connections to Mathematical Practices

Guiding questions are designed to create a framework for the given standards and to engage students in exploring, explaining, and expanding their understanding of the content standards provided in the 2019 *Alabama Course of Study: Mathematics*. Therefore, each guiding question is written to help educators convey important concepts within the standard. By utilizing guiding questions, educators are engaging students in investigating, analyzing, and demonstrating knowledge of the underlying concepts reflected in the standard. An emphasis is placed on the integration of the eight Student for Mathematical Practices.

The Student Mathematical Practices describe the varieties of expertise that mathematics educators at all levels should seek to develop in their students. They are based on the National Council of Teachers of Mathematics process standards and the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up: Helping Children Learn Mathematics*.

The Student Mathematical Practices are the same for all grade levels and are listed below.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Each guiding question includes a representative set of sample activities and examples that can be used in the classroom. The set of activities and examples is not intended to include all the activities and examples that would be relevant to the standard.

Key Academic Terms

These academic terms are derived from the standards and are to be incorporated into instruction by the educator and used by the students.

Additional Resources

Additional resources are included that are aligned to the standard and may provide additional instructional support to help students build toward mastery of the designated standard. Please note that while every effort has been made to ensure all hyperlinks are working at the time of publication, web-based resources are impermanent and may be deleted, moved, or archived by the information owners at any time and without notice. Registration is not required to access the materials aligned to the specified standard. Some resources offer access to additional materials by asking educators to complete a registration. While the resources are publicly available, some websites may be blocked due to Internet restrictions put in place by a facility. Each facility's technology coordinator can assist educators in accessing any blocked content. Sites that use Adobe Flash may be difficult to access after December 31, 2020, unless users download additional programs that allow them to open SWF files outside their browsers.

Printing This Document

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1

Operations and Algebraic Thinking
Write and interpret numerical expressions.
1. Write, explain, and evaluate simple numerical expressions involving the four operations to solve up to two-step problems. Include expressions involving parentheses, brackets, or braces, using commutative, associative, and distributive properties.

Guiding Questions with Connections to Mathematical Practices:

What do parentheses, brackets, or braces mean when used in an expression?

M.P.6. Attend to precision. Describe that grouping symbols can be used to indicate order or designate any expression inside the grouping symbols as a single quantity. For example, the parentheses in the expression $3 \times (6 + 2 + 12)$ mean multiply 3 by the quantity of $6 + 2 + 12$. The numbers can be added first and then multiplied, as in 3×20, or the multiplication can be distributed over the addition, as in $3 \times 6 + 3 \times 2 + 3 \times 12$. Additionally, grouping symbols in expressions can be parentheses ( ), brackets [ ], or braces { }. Further, when a grouping symbol is inside another grouping symbol, like $[8 - (4 + 2)] \div 3$, the operations inside the innermost grouping symbol are carried out first.

- Ask students to consider an expression without grouping symbols, and ask them to insert a set of parentheses that changes the order of operations. Record the expressions and the resulting values of each. For example, $5 \times 4 - 2 + 8 \div 2$ could become $5 \times (4 - 2) + 8 \div 2$, which has a value of 14; or, it could become $5 \times 4 - (2 + 8) \div 2$, which has a value of 15. Point out to students that sometimes the parentheses will group a single operation, like in the expression $5 \times (4 - 2) + 8 \div 2$, and sometimes they will group more than one operation, like in the expression $5 \times (4 - 2 + 8) \div 2$. In these situations, the order of operations should then be applied to the operations within the grouping symbols.

- Ask students to consider a scenario where an operation or series of operations can be recorded as a single quantity. For example, give students the situation “There are 3 packages of markers. Each package of markers has 2 red markers and 4 blue markers.” Explain that 2 red markers and 4 blue markers can be added together because they are both types of markers and, therefore, can be grouped together using parentheses. The total number of markers is the value of the expression $3 \times (2 + 4)$.

How can parentheses, brackets, or braces be used to write and evaluate expressions?

M.P.2. Reason abstractly and quantitatively. Write and solve expressions containing grouping symbols, keeping in mind the meaning of those symbols, the properties of operations, and the convention of the order of operations. For example, $6 \times (4 + 5) \div 2$ can be rewritten as the expression $6 \times 9 \div 2$, which is equivalent to $54 \div 2$, or 27. Additionally, the expression $18 - [2 \times (16 - 12)]$ is equivalent to $18 - (2 \times 4)$, which is equivalent to $18 - 8$, or 10.

- Ask students to write expressions with grouping symbols as equivalent expressions without grouping symbols. Some examples are shown.
    - $5 \times (8 - 3)$

      Subtract 3 from 8 to find the equivalent expression 5×5.
    - $6 - (3 + 1) + 5$

      Add 3 and 1 to find the equivalent expression $6 - 4 + 5$.
    - $2 + [(9 - 2) - 4] \times 3$

      First, subtract 2 from 9 to find the equivalent expression $2 + (7 - 4) \times 3$.

      Then, subtract 4 from 7 to find the equivalent expression $2 + 3 \times 3$.

- Ask students to evaluate expressions by first writing equivalent expressions without grouping symbols and then applying the order of operations to find the value of the expression. Some examples are shown.
  - $[5 + (10 - 4) \times 3] \times 2$

    First, subtract 4 from 10 to find the equivalent expression $(5 + 6 \times 3) \times 2$.

    Then, multiply 6 and 3 to find the equivalent expression $(5 + 18) \times 2$.

    Then, add 5 and 18 to find the equivalent expression 23×2.

    Finally, determine the value of the expression is 46.
  - $24 + [7 - (6 + 1)] \div 2$

    First, add 6 and 1 to find the equivalent expression $24 + (7 - 7) \div 2$.

    Then, subtract 7 from 7 to find the equivalent expression $24 + 0 \div 2$.

    Since $0 \div 2$ is 0, the value of the expression is $24 + 0$, or 24.

When is it necessary to use grouping symbols?

M.P.3. Construct viable arguments and critique the reasoning of others. Demonstrate when grouping symbols are necessary or unnecessary for an expression. For example, for the expression $10 + 5 \div 5$ to equal 3, parentheses need to be placed around $10 + 5$. For the same expression to equal 11, parentheses could be placed around $5 \div 5$, but they are not necessary due to the convention of order of operations. Additionally, the context of the situation that a numerical expression represents may determine if grouping symbols are necessary.

- Ask students to place grouping symbols into an expression to make as many different values as possible. For example, possible student responses for the expression $5 \times 4 + 6 \div 2$ are shown. Note that the original expression, without grouping symbols, is equal to 23, and use this fact to help identify responses for which the grouping symbols are unnecessary.
  - $5 \times (4 + 6) \div 2$

    $5 \times 10 \div 2$, which equals 25
  - $5 \times 4 + (6 \div 2)$

    $5 \times 4 + 3$, which equals 23 (grouping symbols unnecessary)
  - $(5 \times 4 + 6) \div 2$

    $26 \div 2$, which equals 13
  - $5 \times (4 + 6 \div 2)$

    5×7, which equals 35
  - $5 \times [(4 + 6) \div 2]$

    $5 \times [10 \div 2]$

    5×5, which equals 25
- Ask students to place grouping symbols into an expression to represent the language in the context of a problem and make the expression match what is being stated. For example, some situations are shown, with possible expressions to represent each situation.
  - 3 times as much as the sum of 1,402 and 587

    $3 \times (1{,}402 + 587)$
  - 54 plus the product of 12 and the difference of 16 and 10, all divided by 3

    $\{54 + [12 \times (16 - 10)]\} \div 3$
  - the perimeter of a 3-inch by 4-inch rectangle after each side is decreased by 0.5 inch

    $2 \times [(3 - 0.5) + (4 - 0.5)]$

How can a calculation in the form of a written or verbal description be written as a numerical expression?

M.P.6. Attend to precision. Construct an expression from a written or verbal description that uses mathematical vocabulary. For example, write “Add 6 to the product of 8 and 3” as the expression $8 \times 3 + 6$ or $6 + 8 \times 3$ or another equivalent form. Additionally, use the operations of addition and subtraction to represent expressions such as “more than” or “less than.”

- Ask students to write a mathematical expression to represent a given verbal description of a calculation with numbers, and vice versa. Students should create a chart to organize their work.

Verbal Description	Mathematical Expression
divide the sum of five and seventeen in half	$(5 + 17) \div 2$
add 6 to the product of 9 and 2	$6 + 9 \times 2$
eight less than the quotient of sixty and three	$(60 \div 3) - 8$

  Ask students to work together to complete the missing information in the chart. Encourage students to try to write more than one expression to represent a verbal description. Similarly, encourage students to write more than one verbal description to represent an expression. Ask students to share their answers for each description or expression. Record student answers in a class chart and discuss the varieties of answers.

- Ask students to identify relevant information and write a mathematical expression given a verbal description in context. For example, “One rain barrel contains 26 gallons of water and another contains 34 gallons. The contents of the two barrels are combined and then split evenly between the two barrels.” Students should write a mathematical expression representing the amount of water in each barrel, which would be $(26 + 34) \div 2$ gallons.

How can expressions be interpreted without calculation?

M.P.2. Reason abstractly and quantitatively. Interpret, compare, and reason about the meaning of an expression. For example, explain that $(\frac{1}{3} + \frac{5}{6}) - 1$ is 1 less than the sum of $\frac{1}{3}$ and $\frac{5}{6}$. Additionally, know that the order of operations involved affects the wording of the description.

- Ask students to interpret a mathematical expression using the vocabulary of operations. For example, the expression $(27 - 6) \div 3$ means "the difference of twenty-seven and six, divided by three," rather than "the quotient of six and three, subtracted from twenty-seven."
- Ask students to compare the values of two expressions without doing calculations. For example, it can be reasoned that the expressions $\frac{1}{2} \times (3 + 13)$ and $(13 + 3) \div 2$ are equal because multiplying by one-half results in the same value as dividing by 2; also, it can be reasoned that the expression $15 \times (82 - 19)$ is greater in value than the expression $12 \times (82 - 19)$ because the multiple of 15 results in a greater value than the multiple of 12.
- Ask students to reason about the value of an expression without having to calculate. For example, it can be reasoned that the value of $(57 + 198) \div 2$ is half as much as $57 + 198$.

Key Academic Terms:

parentheses, expression, equation, evaluate, grouping symbols, order of operations, braces, brackets, calculate, interpret, compare, reason, equivalent

Additional Resources:

- Video: Numerical expressions
- Lesson: Please excuse my dear Aunt Sally
- Lesson: Interpreting numerical expressions
- Lesson: Writing numeric expressions

2a

Operations and Algebraic Thinking
Analyze patterns and relationships.
2. Generate two numerical patterns using two given rules and complete an input/output table for the data. a. Use data from an input/output table to identify apparent relationships between corresponding terms.

Guiding Questions with Connections to Mathematical Practices:

How can rules be used to generate patterns?

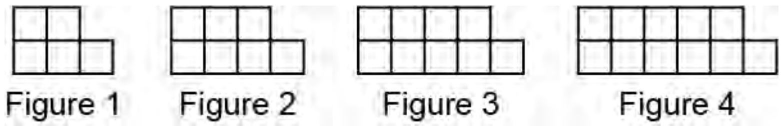
M.P.7. Look for and make use of structure. Generate and describe number patterns when given a rule. For example, given the rule "Add $\frac{1}{2}$" and the starting number 3, generate the list 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$ . . . and describe the pattern. Additionally, given the rule "Multiply by 2" and the starting number 5, generate the list 5, 10, 20, 40 . . . and describe the pattern.

- Ask students to generate a pattern given a rule. For example, given an input pattern with a starting number of 1 and a rule of "Add 1" and an output pattern with a starting number of 10 and a rule of "Add 0.25," students should generate the following table.

Input	Output
1	10
2	10.25
3	10.5
4	10.75
5	11

The patterns can also be listed. For example, the output pattern in a list is 10, 10.25, 10.5, 10.75, 11 . . . and can be described as "increasing by 0.25 each step."

- Ask students to identify a pattern and a rule from a series of numbers or objects with value. For example, given the following series of block figures, students should identify the pattern as increasing by 2 each time and create the rule "Start with 5 and add 2."



Key Academic Terms:

number pattern, generate, sequence, input, output

Additional Resources:

- Lesson: [Generating two numerical patterns]
- Video: [Input-output tables]

2b

Operations and Algebraic Thinking
Analyze patterns and relationships.
2. Generate two numerical patterns using two given rules and complete an input/output table for the data. b. Form ordered pairs from values in an input/output table.

Guiding Questions with Connections to Mathematical Practices:

How are two lists generated by patterns used to create ordered pairs?

M.P.4. Model with mathematics. Given two lists of terms generated by following rules to establish number patterns, create a list of ordered pairs. For example, the list 0, 2, 4, 6, 8, 10 as x-values and the list 0, 3, 6, 9, 12, 15 as y-values would form the ordered pairs (0, 0), (2, 3), (4, 6), (6, 9), (8, 12), and (10, 15). Additionally, the set of ordered pairs (3, 4), (5, $4\frac{1}{2}$), (7, 5), (9, $5\frac{1}{2}$), and (11, 6) can be generated from describing a list of x-values as “start at 3 and add 2” and a list of y-values as “start at 4 and add $\frac{1}{2}$.”

- Ask students to create a table and ordered pairs given two lists, one for *x*-values and one for *y*-values. For example, in the context of recording the number of books a student reads each week over the summer, give students the *x*-values of 1, 2, 3, 4, and 5 weeks and the *y*-values of 2, 4, 6, 8, and 10 books. Then, create a table, as shown.

x	*y*
1	2
2	4
3	6
4	8
5	10

The ordered pairs created from the table are (1, 2), (2, 4), (3, 6), (4, 8), and (5, 10).

- Ask students to create two lists given a set of ordered pairs and to describe the pattern in each list. For example, given the ordered pairs (1, 4), (2, 5), (4, 6), (8, 7), and (16, 8), create a table showing a list of *x*-values that double, starting at 1, and a list of *y*-values that start at 4 and increase by 1.

x	*y*
1	4
2	5
4	6
8	7
16	8

- Ask students to find missing values for *x* and *y* given a table. For example, the table shown is missing an *x*-value and a *y*-value. Find the missing values by first determining the pattern in each column and then applying the patterns.

x	*y*
1	6
2	12
3	24
4	48
5	96

The missing *x*-value is 3 because the pattern for the *x*-values is to add one each time. The missing *y*-value is 48 because the pattern for the *y*-values is to multiply by two each time.

- Ask students to create a set of ordered pairs such that the relationship between the terms fits a specified pattern. For example, given a relationship described as "the *y*-coordinate is always $1\frac{1}{2}$ times the *x*-coordinate, and the difference between successive *x*-coordinates is 2," multiple sets of ordered pairs may fit the description. Two possible student responses are shown.
    - (0, 0), (2, 3), (4, 6), (6, 9) . . .
    - $(1, 1\frac{1}{2})$, $(3, 4\frac{1}{2})$, $(5, 7\frac{1}{2})$, $(7, 10\frac{1}{2})$ . . .

Key Academic Terms:

number pattern, input, output, ordered pairs, generate, sequence

Additional Resources:

- Lesson: Generating two numerical patterns
- Video: Graphing input-output tables

2c

Operations and Algebraic Thinking
Analyze patterns and relationships.
2. Generate two numerical patterns using two given rules and complete an input/output table for the data. c. Graph ordered pairs from an input/output table on a coordinate plane.

Guiding Questions with Connections to Mathematical Practices:

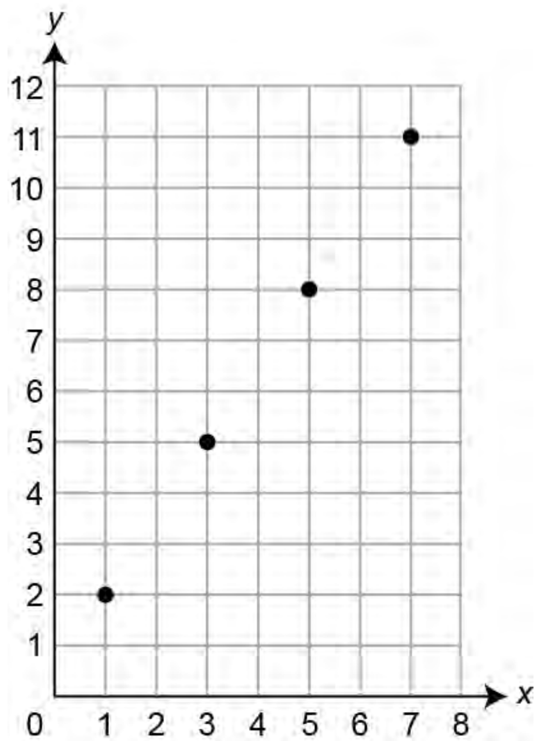
What are the relationships between corresponding terms in coordinate pairs generated by different patterns?

M.P.1. Make sense of problems and persevere in solving them. Identify the additive and multiplicative relationships between corresponding terms. For example, given the ordered pairs (0, 0), (3, 6), (6, 12), and (9, 18), identify that the multiplicative relationship between the corresponding y-coordinate and x-coordinate is the same for every (x, y) coordinate pair. The y-coordinate is 2 times as great as the x-coordinate for every pair. However, the difference between the corresponding x- and y-coordinate pairs changes, following the pattern 0, 3, 6, 9, etc. A graph of the coordinate pairs is points in a straight line, with the y-coordinate always 2 times as great as the x-coordinate. Notice by looking down the columns in a table of x- and y-values that whenever 3 is added to x, 6 is added to y.

x	y
0	0
3	6
6	12
9	18

Additionally, note that the y-value is always 2 times as great as the x-value when looking across the rows of the table.

- Ask students to determine the relationship between corresponding terms in a set of coordinate pairs and describe the pattern that appears when the pairs are graphed on a coordinate plane as points. For example, given the ordered pairs (1, 2), (3, 5), (5, 8), and (7, 11), observe that the difference between the corresponding numbers is not the same for each coordinate pair but progresses in a pattern of 1, 2, 3, 4, and the points fall in a straight line on the graph.



Key Academic Terms:

numerical pattern, graph, coordinate plane, *x*-axis, *y*-axis, origin, *x*-coordinate, *y*-coordinate, ordered pairs, generate, sequence

Additional Resources:

- Lesson: Generating two numerical patterns
- Video: Graphing input-output tables

3a

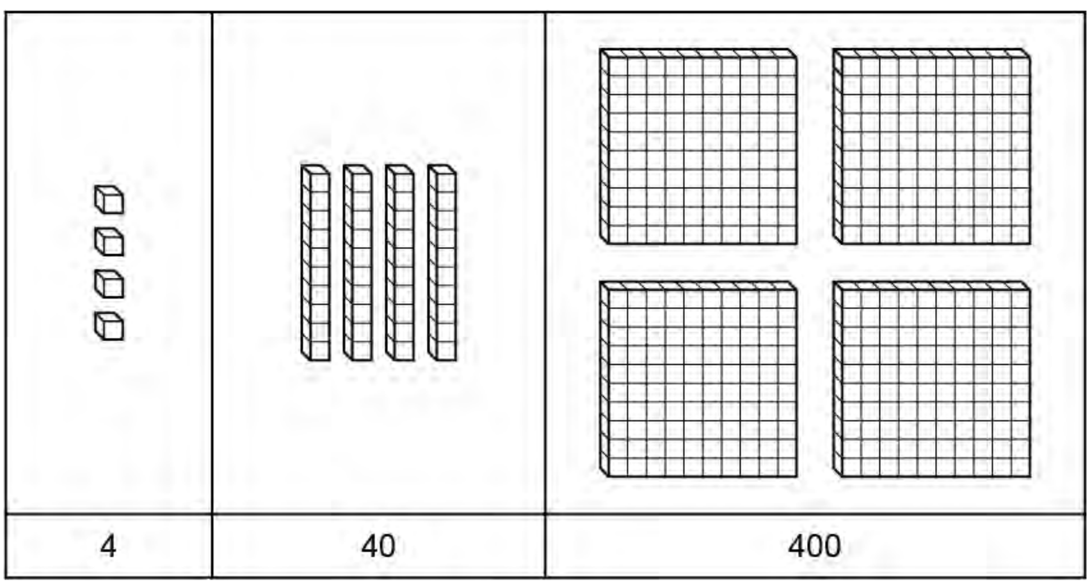
Operations with Numbers: Base Ten
Understand the place value system.
3. Using models and quantitative reasoning, explain that in a multi-digit number, including decimals, a digit in any place represents ten times what it represents in the place to its right and $\frac{1}{10}$ of what it represents in the place to its left. a. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, using whole-number exponents to denote powers of 10.

Guiding Questions with Connections to Mathematical Practices:

How does the value of a digit change in relation to its position within a number?

M.P.8. Look for and express regularity in repeated reasoning. In a multi-digit number, the place value to the left of a given place value is 10 times as much as the given place value. The place value to the right of a given place value is $\frac{1}{10}$ times as much as the given place value. For example, the value of 3 in the number 3.0 is 10 times as much as the value of 3 in 0.3 (i.e., $3.0 = 10 \times 0.3$). The value of 4 in 4.0 is $\frac{1}{10}$ times as much as the value of 4 in 40 (i.e., $4.0 = \frac{1}{10} \times 40$). Additionally, use base-ten blocks or drawings to visualize why a digit in one place represents 10 times as much as it represents in the place to its right.

- Provide students with a number that has a missing digit and ask them to identify what the digit must be so that it represents either 10 times as much as what it represents in the place to the right or $\frac{1}{10}$ times as much as what it represents in the place to the left. For example, given the number 5 ? 3, identify that if the missing digit is 3, then its value is 10 times the value of the 3 in the ones place. If the missing digit is 5, then its value is $\frac{1}{10}$ times the value of the 5 in the hundreds place. Support students with the following sentence frames.
  - The number ____ in the tens place is 10 times as great as the number ____ in the ones place.
  - The number ____ in the tens place is $\frac{1}{10}$ as great as the number ____ in the hundreds place.
- Provide students with base-ten blocks and a multiple of 111. Ask students to represent the value of each digit separately. Then, ask students to compare the numbers of unit blocks that represent each of the digits, noting that there are ten times as many unit blocks representing the tens digit as representing the ones digit and ten times as many unit blocks representing the hundreds digit as representing the tens digit. For example, given the number 444, students could represent the digits as shown.

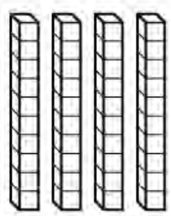
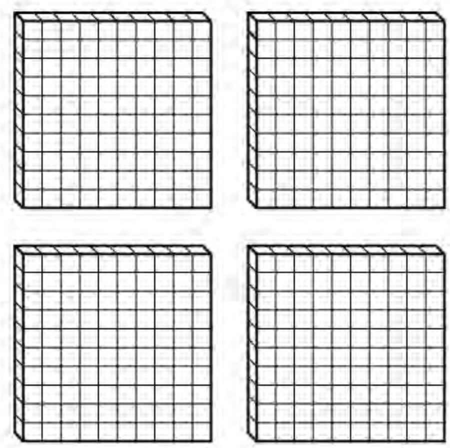


Observe that it would take ten times as many blocks in the ones representation to equal the number of blocks in the tens representation and tens times as many blocks in the tens representation to equal the number of blocks in the hundreds representation.

What patterns become evident when multiplying by powers of 10?

M.P.8. Look for and express regularity in repeated reasoning. Observe that each time a number is multiplied by 10, each digit becomes ten times as great and therefore shifts one place to the left, with zeros used as placeholders if necessary. For example, in the multiplication equation $34 \times 10 \times 10 = 3{,}400$, the digit 3 becomes 10 times as great and then 10 times as great again, so the value changes from the tens place to the thousands place. Additionally, use base-ten blocks to illustrate emerging patterns when multiplying by a power of ten.

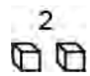
- Ask students to write multiplicative comparisons of the digits in multiples of 111 using exponents. For example, ask students to extend the table in the previous example.

		
4	40	400
4	4 × 10	4 × 100
4	4×10^1	4×10^2

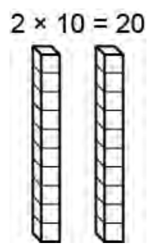
  Recognize that multiplying a single digit by a power of 10 increases the place value of the digit a number of times equal to the exponent.

- Ask students to evaluate expressions and describe patterns of a number multiplied by a power of 10. Write three or more expressions that multiply a value by different powers of 10. For example:
  - $4 \times 10 = 40$ (the digit 4 moves left one place)
  - $4 \times 10 \times 10 = 400$ (the digit 4 moves left two places)
  - $4 \times 10 \times 10 \times 10 = 4{,}000$ (the digit 4 moves left three places)

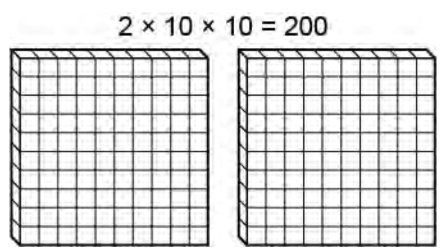
- Ask students to show patterns of a number multiplied by a power of 10 using base-ten blocks or other visual representations.
    - First, represent the number 2 with base-ten blocks.



    - Then, represent 2×10 by creating the original representation 10 times. Note the number of blocks.



    - Next, represent $2 \times 10 \times 10$ by creating the second representation 10 times. Note the number of blocks.



    - Finally, identify the pattern in the total number of blocks each time the original number is multiplied by 10 (i.e., 2, 20, 200 . . .).

Key Academic Terms:

digit, place value, power of ten

Additional Resources:

- Lesson: [1/10 of . . .]
- Lesson: [Powers of ten]

3b

Operations with Numbers: Base Ten
Understand the place value system.
3. Using models and quantitative reasoning, explain that in a multi-digit number, including decimals, a digit in any place represents ten times what it represents in the place to its right and $\frac{1}{10}$ of what it represents in the place to its left. b. Explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10, using whole-number exponents to denote powers of 10.

Guiding Questions with Connections to Mathematical Practices:

How can place value understanding be used to determine the location of a decimal point?

M.P.2. Reason abstractly and quantitatively. Determine the value of a product or quotient of an expression involving a power of 10 by observing that each power of 10 multiplies or divides the value of each of the digits by 10. Zeros are used as placeholders between the digits and the decimal point to show when there is no value for that location. For example, $0.7 \times \frac{1}{10}$ will cause the digit 7 to change in value from the tenths to the hundredths place, and there will be no tenths. Therefore, it results in the answer 0.07. Additionally, use place value tables to visualize the value of a digit and the location of the decimal point when a number is multiplied by a power of 10.

- Ask students to evaluate a number when it is multiplied or divided by 1,000, 100, 10, or 1. For example, ask students to evaluate the pattern created in the table shown.

$5.0 \times 1{,}000$	$=$	$5{,}000.0$
5.0×100	$=$	500.0
5.0×10	$=$	50.0
5.0×1	$=$	5.0
$5.0 \div 10 = 5.0 \times \frac{1}{10} = \frac{5.0}{10}$	$=$	0.5
$5.0 \div 100 = 5.0 \times \frac{1}{100} = \frac{5.0}{100}$	$=$	0.05
$5.0 \div 1{,}000 = 5.0 \times \frac{1}{1{,}000} = \frac{5.0}{1{,}000}$	$=$	0.005

Have students use the pattern above to evaluate the product of other numbers and a power of 10. For example, given the expressions $2 \times \frac{1}{10}$, $3.4 \times 1{,}000$, and $5.6 \times \frac{1}{100}$, students should be able to determine the products 0.2, 3,400, and 0.056.

- Ask students to write a number with at least one nonzero digit on each side of the decimal point and complete a decimal table. Then, multiply the number by 10 by multiplying the value of each digit by 10. Record the product on the decimal table. Then, multiply the original number by 100 and 1,000. Record the products on the decimal table. Explain how each digit changes in value depending on the power of 10 by which it is multiplied. For example, ask students to use 6.23 to create a decimal table like the one shown.

	Thousands	Hundreds	Tens	Ones	.	Tenths	Hundredths
6.23				6	.	2	3
6.23 × 10			6	2	.	3	0
6.23 × 100		6	2	3	.	0	0
6.23 × 1,000	6	2	3	0	.	0	0

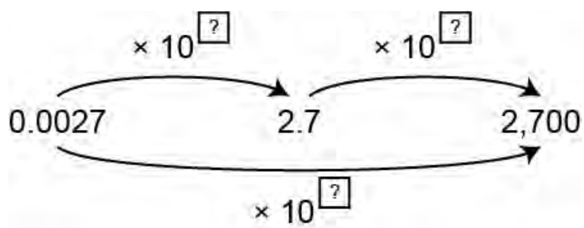
  In the first multiplication expression, each digit is 10 times as great as it was and is therefore 1 place to the left of where it was originally. In the second multiplication expression, each digit is 100 times as great as it was and is therefore 2 places to the left of where it was originally. In the third multiplication expression, each digit is 1,000 times as great as it was and is therefore 3 places to the left of where it was originally.

How is the exponent for a power of 10 related to the placement of a decimal point?

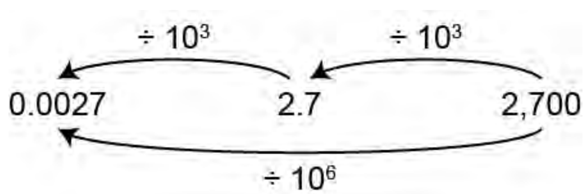
M.P.8. Look for and express regularity in repeated reasoning. Interpret the exponent of a power of 10 as an indication of how many times 10 is being multiplied or divided, change the value of each of the digits being multiplied accordingly to determine the decimal placement, and use zeros to show the locations of place values when there is no value given. For example, $10^3 = 10 \times 10 \times 10$ which is 1,000, so the exponent of 3 indicates that all the digits being multiplied by 10^3 will be multiplied by 1,000. Determine and create patterns that are based on products of powers of 10. Additionally, $\frac{1}{10^3} = \frac{1}{10 \times 10 \times 10}$ which is $\frac{1}{1,000}$, so the exponent of 3 when used with division indicates that all the digits being divided by 10^3 will be divided by 1,000.

- Ask students to write provided numbers or expressions in exponential form. For example, given the expressions $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$, $10 \times 10 \times 10 \times 10$, 1,000, and 100,000, write 10^7, 10^4, 10^3, and 10^5, respectively. Additionally, given the expressions $\frac{1}{1{,}000}$, $\frac{1}{10{,}000}$, $\frac{1}{10 \times 10 \times 10 \times 10}$, and $\frac{1}{10 \times 10 \times 10 \times 10 \times 10 \times 10}$, write $\frac{1}{10^3}$, $\frac{1}{10^4}$, $\frac{1}{10^4}$, and $\frac{1}{10^6}$, respectively.

- Ask students to identify what each term in a given pattern is multiplied by to obtain the subsequent term. Express each answer as a power of 10. For example, given the number pattern 0.0027, 2.7, 2,700, . . ., identify that each term is multiplied by 10^3 to obtain the subsequent term.



Notice that when each number is multiplied by 10^3, the result contains the exact same nonzero digits as the original number, but the location of the decimal point is three places farther to the right. Moreover, when looking at a number two terms ahead in this pattern, the first term needs to be multiplied by 10^3 twice. This is equivalent to multiplying by 10 six times, so the location of the decimal point of the number two terms later is six places farther to the right.



Additionally, notice that the sequence can be reversed to illustrate that dividing by a power of 10 results in a decimal point that is farther to the left.

Key Academic Terms:

product, multiply, divide, power of 10, decimal, decimal point, exponent, append, factor, base, expression, place value

Additional Resources:

- Activity: Multiplying decimals by 10, 100 & 1,000
- Video: Powers of 10
- Lesson: Powers of ten applications
- Activity: Dividing decimals by 10, 100 or 1,000

4a

Operations with Numbers: Base Ten
Understand the place value system.
4. Read, write, and compare decimals to thousandths. a. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form. *Example*: 347.392 = $3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times (\frac{1}{10}) + 9 \times (\frac{1}{100}) + 2 \times (\frac{1}{1000})$.

Guiding Questions with Connections to Mathematical Practices:

How can place value understanding help to read decimal numbers?

M.P.6. Attend to precision. Observe the placement of digits in a decimal number to determine how to read the number. For example, 3.042 is read as "three and forty-two thousandths" because there is a 3 in the ones place, a 0 in the tenths place, a 4 in the hundredths place, and a 2 in the thousandths place. Additionally, write a number in standard form when it is read out loud or written in words.

- Ask students to identify the place value of the last digit of a given number. Students should then use the named place value as a guide to say and write the number. For example, given the number 24.68, students should identify that the 8 is in the hundredths place. A place value table can be helpful, like the one shown.

Tens	Ones	.	Tenths	Hundredths
2	4	.	6	8

  Students should then practice saying and writing the number as "twenty-four and sixty-eight hundredths."

- Ask students to read numbers that have the same digits in different places. Give students two decimal numbers with similar digits, such as 1.35 and 1.305. Students should then read both numbers out loud and explain to a partner how they decided to say each number. Students should know that 1.35 can be read as "one and thirty-five hundredths" because the last digit is in the hundredths place, and that 1.305 can be read as "one and three hundred five thousandths" because the last digit is in the thousandths place.

- Given a number written in word form, ask students to write the number in standard form. Give students a number, such as “nine and eight hundred twenty-one thousandths.” Ask students to underline or say the place value of the last digit. Students should then use the designated place to help them write the number in standard form: 9.821. Students may be given numbers orally, written in word form, or both.

How can place value understanding help to write decimals in expanded form?

M.P.6. Attend to precision. Represent decimal numbers in expanded form with single digits multiplied by the power of 10 appropriate to place value. For example, the expanded form of 13.94 equals $1 \times 10 + 3 \times 1 + 9 \times \frac{1}{10} + 4 \times \frac{1}{100}$ because the 1 is in the tens place, the 3 is in the ones place, the 9 is in the tenths place, and the 4 is in the hundredths place. Additionally, use decimal tables to help write decimals in expanded form.

- Ask students to write a given number in a decimal table and then in expanded form. Give students a decimal number, such as 203.654. Students then record the digits for each place in a decimal table.

Hundreds	Tens	Ones	.	Tenths	Hundredths	Thousandths
2	0	3	.	6	5	4

Next, students use the information in the decimal table to write the number in expanded form. Move from left to right, multiplying each digit by its place value and then recording the number as the sum of the products.

$$2 \times 100 + 0 \times 10 + 3 \times 1 + 6 \times \frac{1}{10} + 5 \times \frac{1}{100} + 4 \times \frac{1}{1{,}000}$$

- Ask students to write numbers in expanded form when given the number in written or standard form. Give students a number in written form, such as “ninety-eight and sixteen-hundredths.” Students then record the number in expanded form: $9 \times 10 + 8 \times 1 + 1 \times \frac{1}{10} + 6 \times \frac{1}{100}$. Some students may write the number in standard form first in order to help them write the number in expanded form. Also, give students a number written in standard form, such as 22.222. Students then record the number in expanded form: $2 \times 10 + 2 \times 1 + 2 \times \frac{1}{10} + 2 \times \frac{1}{100} + 2 \times \frac{1}{1{,}000}$. The numbers can be written and/or given orally. Ask students to discuss the processes they used to change numbers from one form to another.

Key Academic Terms:

decimal, thousandths, hundredths, tenths, base-ten, expanded form, place value, power of 10, product, standard form

Additional Resources:

- Video: Comparing decimals song
- Lesson: Reading and writing with decimals

4b

Operations with Numbers: Base Ten
Understand the place value system.
4. Read, write, and compare decimals to thousandths. b. Compare two decimals to thousandths based on the meaning of the digits in each place, using >, =, and < to record the results of comparisons.

Guiding Questions with Connections to Mathematical Practices:

Which place value strategies can be used to compare values of decimal numbers?

M.P.5. Use appropriate tools strategically. Extend place value understanding from whole numbers to decimals that are in the tenths, hundredths, or thousandths places by using visual models or other place value strategies. For example, know that 7.80 is greater than 7.799 by comparing the tenths place in both numbers and recognizing that $\frac{8}{10}$ is greater than $\frac{7}{10}$. The digits after the tenths place in the number 7.799 can be disregarded for the comparison because all the digits to the right of the tenths place have values that are less than $\frac{8}{10}$ even though the digits are greater than 8. Additionally, use decimal tables as tools for comparing decimals.

- Ask students to create true comparisons of numbers with missing digits. Give students a decimal number along with a second decimal number that has a missing digit. Students then identify all possible digits for the missing digit that makes the second number less than the first, and vice versa. For example, students should compare the two decimal numbers shown.

  $\underline{6}\,\underline{9}\,\underline{8}\,.\,\underline{4}\,\underline{7}\,\underline{1} \qquad \underline{6}\,\underline{9}\,\underline{8}\,.\,\underline{4}\,\underline{?}\,\underline{1}$

  Students should determine that when the missing digit is 0, 1, 2, 3, 4, 5, or 6, the second number will be less than the first number. Also, when the missing digit is 7, the second number will be equal to the first number, and when the missing digit is 8 or 9, the second number will be greater than the first number.

- Ask students to write decimal numbers having similar digits in a decimal table to help compare the numbers. Students record the numbers in the same decimal table and compare corresponding place value digits from left to right. For example, given the numbers 12.415 and 12.355, students record the numbers in the table, as shown.

Tens	Ones	.	Tenths	Hundredths	Thousandths
1	2	.	4	1	5
1	2	.	3	5	5

  Students should observe that all the digits have the same value when moving from left to right until the tenths place. Students should then determine that the number 12.415 is greater than the number 12.355 because four-tenths is greater than three-tenths.

How can decimal numbers be compared using mathematical symbols?

M.P.4. Model with mathematics. Record the results of the comparison of two decimals through thousandths by using the mathematical symbols $>$, $<$, or $=$. For example, $0.65 > 0.645$. Additionally, make comparisons of two numbers by first using the $>$ symbol and then changing the order of the numbers and using the $<$ symbol. For example, the numbers 0.19 and 0.09 can be compared as $0.19 > 0.09$ or $0.09 < 0.19$.

- Ask students to determine the correct mathematical symbol ($>$, $<$, or $=$) needed to create a true comparison of two decimal numbers. Give students pairs of decimal numbers shown side by side. Students then identify and record the symbols to make true comparisons.

  $$1.02 \square 1.11$$

  $$20.658 \square 20.649$$

  $$543.302 \square 543.299$$

  Students should record the correct symbols as $<$, $>$, and $>$. Ask students to explain to a partner how they chose the correct symbols.

- Ask students to create two comparisons for the same pair of numbers. Give students two numbers, such as 5.123 and 5.213. Students then record the first comparison using the $>$ symbol: $5.213 > 5.123$. Next, students record the second comparison using the $<$ symbol: $5.123 < 5.213$.

Key Academic Terms:

compare, decimal, thousandths, hundredths, tenths, symbol, greater than, less than, equal, place value strategies

Additional Resources:

- Lesson: Number and operations in base ten
- Tutorial: Comparing decimals to thousandths
- Video: Ordering decimals
- Lesson: Dewey Decimal system for ordering decimals

5

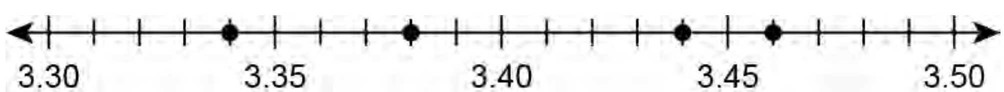
Operations with Numbers: Base Ten
Understand the place value system.
5. Use place value understanding to round decimals to thousandths.

Guiding Questions with Connections to Mathematical Practices:

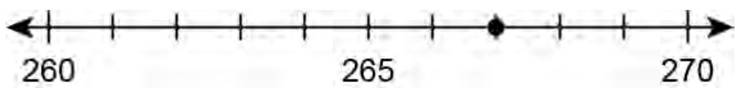
How is rounding to the nearest decimal place similar to rounding whole numbers?

M.P.8. Look for and express regularity in repeated reasoning. Extend learning regarding rounding to the nearest ten, hundred, thousand, etc., to rounding to the nearest tenth, hundredth, thousandth, etc., by using similar strategies. For example, use a number line from 0.4 to 0.5 to determine whether 0.42 is closer to 0.4 or 0.5; this strategy is like using a number line from 40 to 50 to help determine whether 42 is closer to 40 or 50. Additionally, decompose decimal numbers to help find similarities between rounding whole numbers and rounding decimal numbers.

- Ask students to use a number line to help round decimal numbers. Give students a list of decimal numbers, such as 3.34, 3.38, 3.44, and 3.46. Students then plot each decimal number on a number line labeled from 3.30 to 3.50. Ask students to determine which decimal numbers round to 3.40 when rounded to the nearest tenth, based on their locations on the number line. Determine that 3.38 and 3.44 round to 3.40 when rounded to the nearest tenth. Students should also determine that the remaining numbers round to either 3.30 or 3.50. Ask students to use the number line to explain each answer.



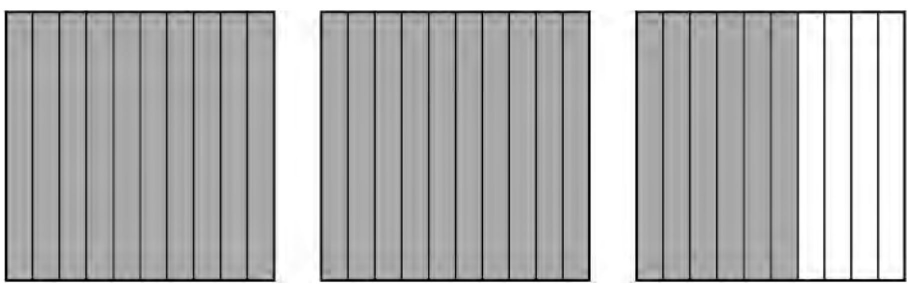
- Ask students to use a number line and the decomposition of numbers to extend the concept of rounding whole numbers to rounding decimal numbers. Give students a number line labeled from 260 to 270 with a point plotted at 267. Then, ask students to round 267 to the nearest ten.



Note that 267 rounds to 270 because 267 is closer to 270 than it is to 260 on the number line. Next, ask students to decompose the number 2.67 and to record the results in a decimal place value table.

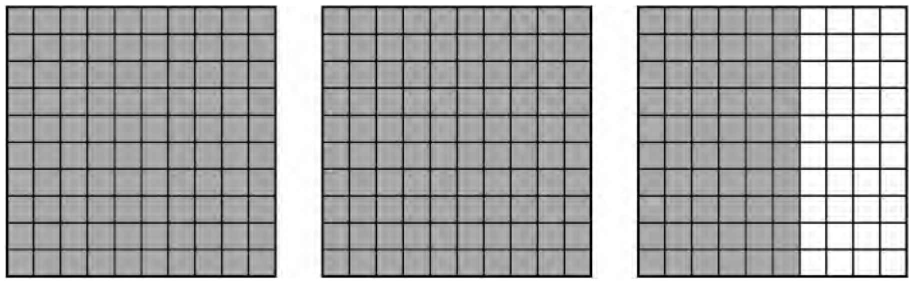
Ones	.	Tenths	Hundredths
2	.	6	7

Note that 2 ones is equivalent to 20 tenths, and ask students to model the total number of tenths, 20 + 6, using a drawing and a table.



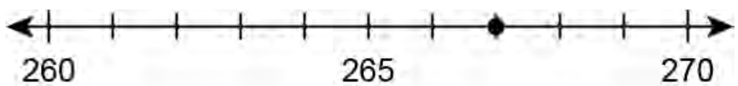
Ones	.	Tenths	Hundredths
	.	20 + 6 = 26	7

Note that 26 tenths is equivalent to 260 hundredths, and ask students to model the total number of hundredths, $260 + 7$, using a drawing and a table.



Ones	.	Tenths	Hundredths
	.		260 + 7 = 267

Students should determine that 2.67 is equivalent to 267 hundredths. This could also be shown using the drawing above and shading an additional 7 small squares to represent the 7 hundredths. Ask students to refer back to 267 on the number line.



Discuss how rounding 267 to the nearest ten is similar to rounding 2.67 to the nearest tenth.

What makes "5" significant when rounding?

M.P.7. Look for and make use of structure. Identify that 5 is significant because it represents the halfway point between two values on a number line. For example, on a number line from 1.7 to 1.8, the interval between 1.7 and 1.8 is divided into ten equally sized sections that are marked with the numbers 1.71, 1.72, 1.73, 1.74, 1.75, 1.76, 1.77, 1.78, and 1.79. The number 1.75 is the same distance from 1.7 as it is from 1.8 on the number line, so it represents the halfway point between 1.7 and 1.8. To the left of 1.75, all the values are closer to 1.7, and to the right of 1.75, all the values are closer to 1.8. The value of 1.75 will round up to 1.8 because half of the values round to 1.7 and half of the values round to 1.8. The value 1.75 is the halfway point, so the digit 5 in the hundredths place is significant when rounding to the nearest tenth. The same is true when rounding to any place value; the digit 5 is significant when rounding to a place value that is 10 times as much as the place value of the 5. Additionally, solve real-world rounding problems that involve numbers containing the digit 5.

- Ask students to round a given number that includes the digit 5 multiple times, such as 5.5555, to different places. Students round the given number to the nearest tenth (5.6), hundredth (5.56), and thousandth (5.556). Students then draw or discuss the process used to round the given number to the different places. Discuss the importance of the digit 5 when rounding.

- Ask students to determine the greatest and least decimal numbers, extended to the tenths place, that will round to a given whole number. For example, give students the number 8. Students then identify 7.5 as the least decimal number extended to the tenths place that will round to 8; and they should identify 8.4 as the greatest decimal number extended to the tenths place that will round to 8. Next, ask students to draw a representation of their solution processes using a number line or some other visual representation.

- Ask students to solve real-world problems that involve rounding decimal numbers containing the digit 5. For example, give students the problem shown.

  A meteorologist records the daily high temperatures of five different cities, in °F (degrees Fahrenheit), as shown.

  65.3, 65.5, 68.5, 70.5, 71.4

  She rounds each temperature to the nearest whole degree for a weather report. What numbers appear in the meteorologist's weather report?

  Students should record the numbers 65, 66, 69, 71, and 71.

Key Academic Terms:

round, place value, tenths, hundredths, thousandths, decimal, number line

Additional Resources:

- Activity: Roll It! Rounding game
- Lesson: Rounding decimals
- Video: Rounding numbers

6

Operations with Numbers: Base Ten
Perform operations with multi-digit whole numbers and decimals to hundredths.
6. Fluently multiply multi-digit whole numbers using the standard algorithm.

Guiding Questions with Connections to Mathematical Practices:

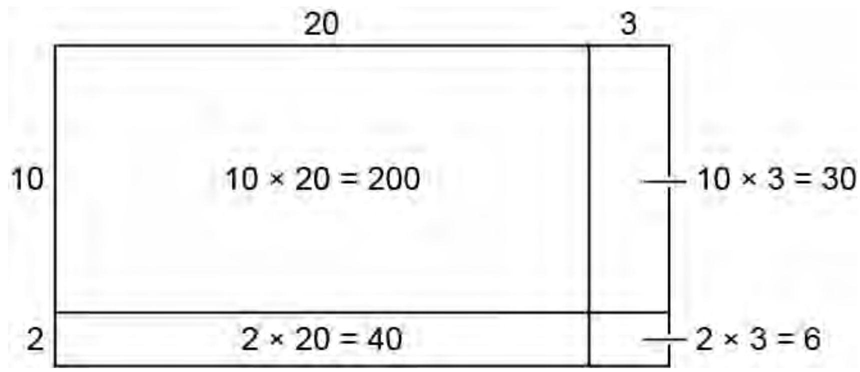
How are multiplication algorithms related to each other?

M.P.8. Look for and express regularity in repeated reasoning. Demonstrate the standard algorithm for multi-digit multiplication and explain how it relates to previously used multiplication methods such as the distributive property, partial products, and area models. For example, show that multiplying 43×17 using the standard algorithm is connected to multiplying and adding $(40 \times 10) + (40 \times 7) + (3 \times 10) + (3 \times 7)$. Additionally, use arrays to model multiplication.

- Ask students to find the product of two given numbers using the standard algorithm. Students then use an area model to find the product of the same two numbers. Next, students identify the multiplication steps represented in their work with the standard algorithm and area model. For example, ask students to multiply the numbers 12 and 23. Find the product using the standard algorithm, as shown.

$$\begin{array}{r} 12 \\ \times\ 23 \\ \hline 36 \\ +\ 240 \\ \hline 276 \end{array}$$

Next, create an area model to represent 12×23, as shown.

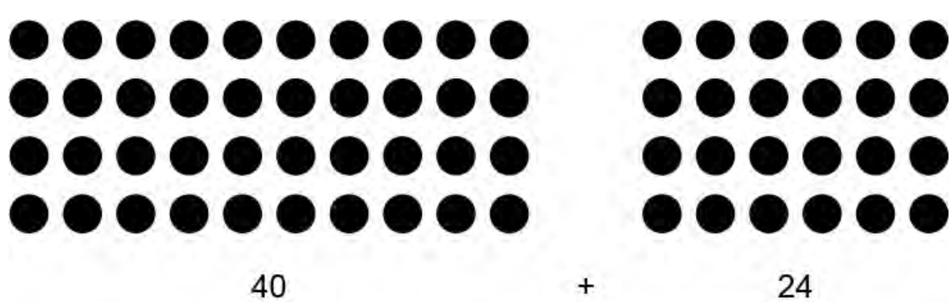


Ask students to identify the multiplication steps represented in both the standard algorithm and in the area model. For example, notice where $10 \times 3 = 30$ occurs in the standard algorithm and in the area model. Students should also discuss how the two methods relate in general. Guide students to see that the sum of 36 and 240 is the same as the sum of $(30 + 6)$ and $(200 + 40)$, so both methods require finding the sum of 30, 6, 200, and 40 to find the product of 12 and 23.

- Ask students to find the product of two given numbers using the distributive property and using the standard algorithm. Students then model the partial products with an array. For example, give students 4×16, which can be decomposed into $4 \times (10 + 6)$. Students then apply the distributive property to get $40 + 24 = 64$. Next, find the product of 4 and 16 using the standard algorithm, as shown.

$$\begin{array}{r} {\scriptstyle 2} \\ 16 \\ \times\ \ 4 \\ \hline 64 \end{array}$$

Then, ask students to represent 4×16 by drawing an array that represents the expression $4 \times (10 + 6)$.



Discuss the similarities between the three methods. Guide students to see that all three methods require finding the sum of 40 and 24.

- Ask students to calculate the same product using both the standard algorithm as well as partial products. For example, ask students to find the product of 364 and 52. Students compute the product as shown.

Standard Algorithm

$$\begin{array}{r} 364 \\ \times\ 52 \\ \hline 728 \\ +\ 18{,}200 \\ \hline 18{,}928 \end{array}$$

Partial Products

$$\begin{array}{rl} 364 & \\ \times\ 52 & \\ \hline 8 & 2 \times 4 \\ 120 & 2 \times 60 \\ 600 & 2 \times 300 \\ 200 & 50 \times 4 \\ 3{,}000 & 50 \times 60 \\ +\ 15{,}000 & 50 \times 300 \\ \hline 18{,}928 & \end{array}$$

Discuss with students the similarities between the two methods. In particular, notice that the 728 in the standard algorithm is the sum of 8 + 120 + 600 in the partial products and that the 18,200 is the sum of 200 + 3,000 + 15,000. Observe that the two methods are almost identical except that the standard algorithm combines multiple rows into a single row.

Key Academic Terms:

multiply, multi-digit, standard algorithm, distributive property, partial products, area model, array

Additional Resources:

- Lesson: Multiply multi-digit numbers
- Activity: Multiplication Pal

7

Operations with Numbers: Base Ten
Perform operations with multi-digit whole numbers and decimals to hundredths.
7. Use strategies based on place value, properties of operations, and/or the relationship between multiplication and division to find whole-number quotients and remainders with up to four-digit dividends and two-digit divisors. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Guiding Questions with Connections to Mathematical Practices:

How does knowledge of place value and properties of operations help solve division problems?

M.P.2. Reason abstractly and quantitatively. Decompose and compose numbers in a variety of ways using place value and the properties of operations to demonstrate different strategies for division, such as partial quotients. This prepares students to learn the standard division algorithm, which is introduced in grade 6. For example, when solving $345 \div 15$, use knowledge of the number of 15s in 300 and the number of 15s in 45 to get $20 + 3 = 23$. Additionally, find quotients by rewriting division problems as corresponding multiplication problems with unknown factors.

- Given a three-digit dividend and a two-digit divisor, ask students to decompose the dividend to find partial quotients. For example, give students a dividend of 864 and a divisor of 16. Students decompose 864 into $800 + 64$. Next, divide each term of the decomposition by the divisor, 16.

  $$800 \div 16 = 50$$

  $$64 \div 16 = 4$$

  Students then add the results: $50 + 4 = 54$. Therefore, $864 \div 16 = 54$. Ask students to draw a representation of the division (e.g., an area model) and then explain their drawing to a partner.

- Given a division problem with a three- or four-digit dividend and a one- or two-digit divisor, ask students to change the division problem into a corresponding multiplication problem. For example, give students the problem 180 ÷ 20 = ?. Students then write a corresponding multiplication problem: ? × 20 = 180. Students can use strategies such as skip-counting to help solve the multiplication problem.

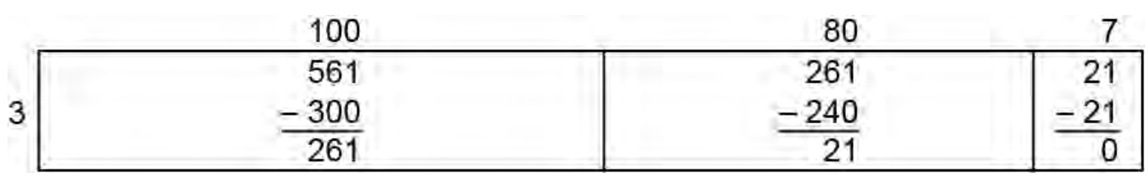
Skip-Count	20	40	60	80	100	120	140	160	180
Number of 20s	1	2	3	4	5	6	7	8	9

Determine that there are 9 “20s” in 180, and so 180 ÷ 20 = 9. As a further example, give students the problem 1,230 ÷ 15 = ?. Students then write a corresponding multiplication problem: ? × 15 = 1,230. Students can reason that 10 × 15 = 150 and therefore 20 × 15 = 300. Because four 300’s total 1,200, 80 × 15 = 1,200. To reach a total of 1,230, an additional 30 is needed. Since 30 = 2 × 15, the complete product is 82 × 15 = 1,230, and so 1,230 ÷ 15 = 82.

How do area models and arrays connect with equations and division strategies?

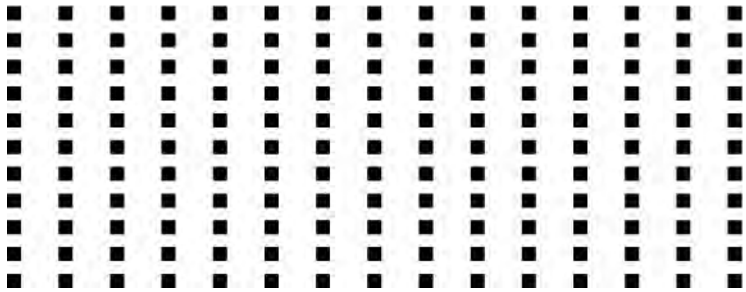
M.P.4. Model with mathematics. Connect area models to equations and division strategies to explain and illustrate a calculation. For example, when solving 1,056 ÷ 22, use an area model of a rectangle with one side length of 22 and an area of 1,056 to show that the unknown length has 4 tens and 8 ones, and the quotient of the division problem is 48. Additionally, use arrays to visualize division problems involving whole number dividends and divisors.

- Ask students to explain a given area model and to write a corresponding division equation. For example, give students the model shown.



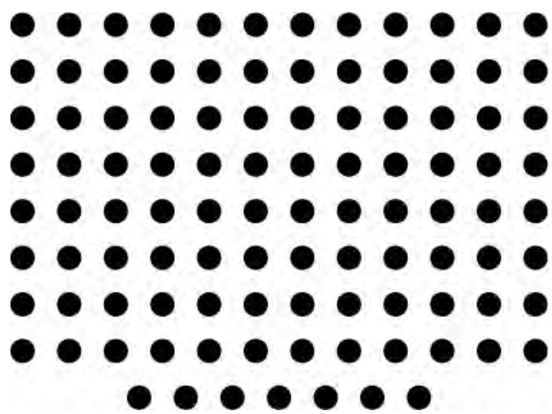
Students may explain that 561 can be divided into 3 groups of 100 with 261 remaining, 261 can be divided into 3 groups of 80 with 21 remaining, and 21 can be divided into 3 groups of 7 with 0 remaining. Next, write the corresponding division equation for the model: 561 ÷ 3 = 187.

- Ask students to explain how an array models a given real-world division problem. For example, “There are 165 folding chairs in a gymnasium. The chairs will be placed in rows of 15 for a concert. How many rows of chairs will be made for the concert?”

  Explain how each part of the problem is modeled in the array. For example, the array includes 165 squares to represent all the chairs. The 15 squares in each row represent one row of chairs. There are 11 columns, which represent the total number of rows of chairs.

- Ask students to use an array model to illustrate the meaning of a remainder. For example, “Theresa has 103 photos to put into an album. Each page of the photo album can hold up to 12 photos. What is the meaning of the quotient and the remainder of the expression $103 \div 12$ in this context?”

  The array model shows that $103 \div 12 = 8$ R7 because 103 dots can be arranged into 8 rows of 12 dots in each row with 7 dots left over. Each row of 12 dots represents one page of 12 photos. The 8 rows represent 8 complete pages of photos. The 7 remaining dots that represent the remainder indicate that an incomplete page of photos is needed.

Key Academic Terms:

quotient, dividend, divisor, divide, multiply, multiple, equation, remainder, area model, decompose, partial quotients, array

Additional Resources:

- Activity: Minutes and days
- Lessons: Grade 5 mathematics module 2

8a

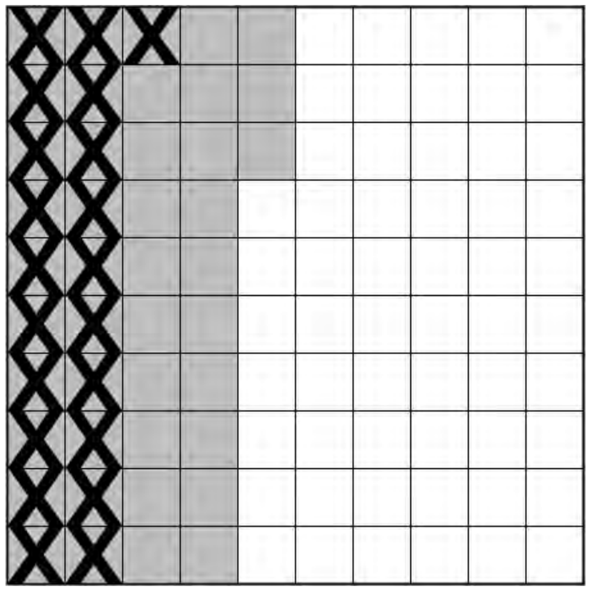
Operations with Numbers: Base Ten
Perform operations with multi-digit whole numbers and decimals to hundredths.
8. Add, subtract, multiply, and divide decimals to hundredths using strategies based on place value, properties of operations, and/or the relationships between addition/subtraction and multiplication/division; relate the strategy to a written method, and explain the reasoning used. a. Use concrete models and drawings to solve problems with decimals to hundredths.

Guiding Questions with Connections to Mathematical Practices:

How does using the four operations with decimal numbers relate to using the four operations with whole numbers?

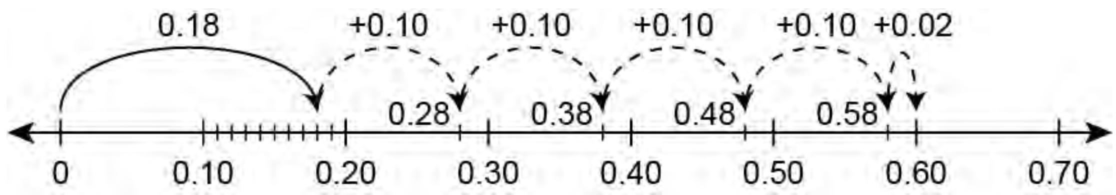
M.P.8. Look for and express regularity in repeated reasoning. Describe the similarities when adding, subtracting, multiplying, and dividing decimal numbers compared to the same operations with whole numbers. For example, using base-ten blocks to multiply 54×3 with the hundred grid representing 100, compare to multiplying 0.54×3 with the hundred grid representing one whole. Additionally, use place value tables to illustrate commonalities between operations with whole numbers and operations with decimal numbers.

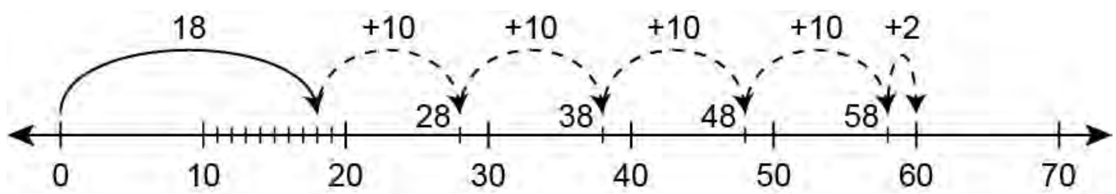
- Provide students with a pair of two-digit whole numbers and ask them to find the sum or difference using any method. Then, provide students with a pair of decimal numbers that contain the same two digits and ask them to find the sum or difference using a 10 by 10 decimal grid (where the entire grid represents 1 whole). Note the similarity between the two answers. For example, ask students to find the difference of 43 and 21 and then check for a correct answer of 22. Then, ask students to find the difference between 0.43 and 0.21 using a 10 by 10 decimal grid by first shading 43 squares and then crossing out 21 of those squares.



Note that 43 – 21 = 22 and 0.43 – 0.21 = 0.22.

- Ask students to find the sum or difference of a pair of two-digit decimals using a number line. Then ask them to use a number line to find the sum or difference of a pair of two-digit whole numbers that contain the same two digits, and ask them to note the similarities in the models. For example, ask students to use a number line to find the sum of 0.18 and 0.42 and compare that number line to a number line that is used to find the sum of 18 and 42.





Note that both number line models have the exact same structure.

- Provide students with a pair of two-digit whole numbers and ask them to write the numbers in a place value table. Then, ask students to determine the sum by adding the values of the tens together and the values of the ones together. Next, provide students with a pair of two-digit decimal numbers that have the same digits as the original pair and ask them to write the numbers in the place value table. Ask students to determine the sum by adding the values of the tenths together and the values of the hundredths together. Note the similarities. For example, ask students to write 56 and 32 in a place value table and find the sum. Then, ask students to do the same thing for 0.56 and 0.32.

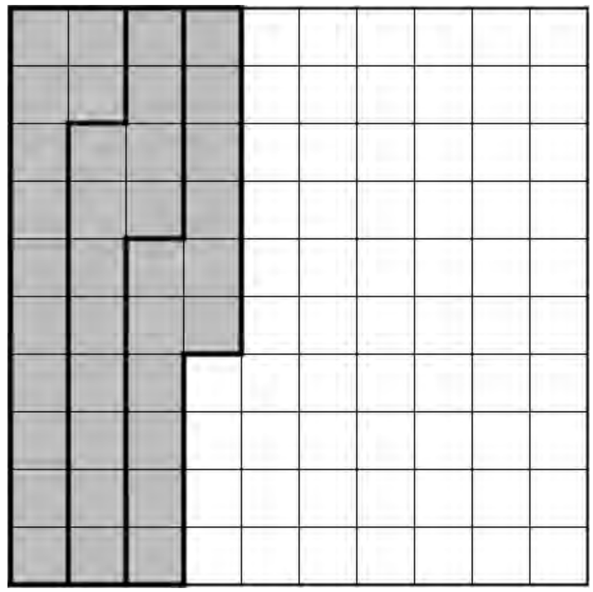
	Tens	Ones	.	Tenths	Hundredths
56	5	6			
32	3	2			
56 + 32	8	8			
0.56		0	.	5	6
0.32		0	.	3	2
0.56 + 0.32		0	.	8	8

Note that $56 + 32 = 88$ (8 tens and 8 ones) and $0.56 + 0.32 = 0.88$ (8 tenths and 8 hundredths).

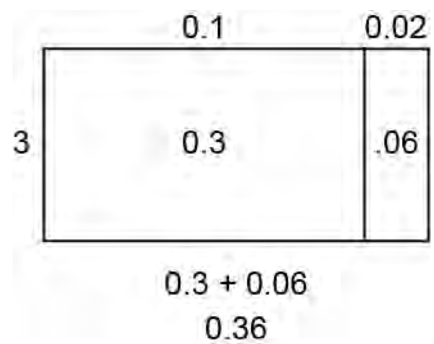
How can problems involving the four operations with decimal numbers be solved in a variety of ways?

M.P.1. Make sense of problems and persevere in solving them. Use a variety of strategies, such as modeling, connecting to fractions, using patterns, and reasoning about the size of a number to solve problems involving the four operations with decimal numbers. For example, when solving $4 \div 0.1$, use the meaning of division and place value to think of the problem as “How many one tenths are in forty tenths?” to find a solution of 40. Additionally, interpret a model that represents an operation with one or more decimal numbers and write a corresponding equation.

- Provide students with a problem involving a decimal number and any one of the four operations. Then, ask students to solve the problem using at least two different methods. For example, begin with the problem 0.12×3. Using a 10 by 10 decimal grid, students could shade 12 squares three times before counting up the total of 36 squares (0.36).

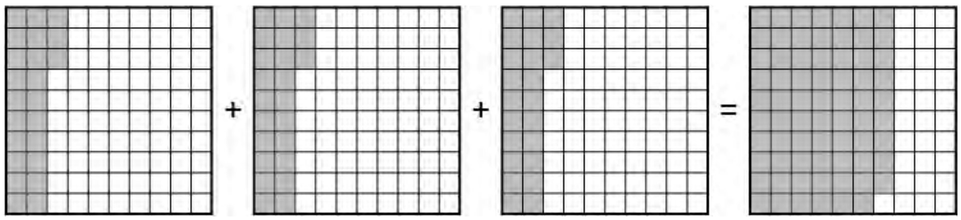


Then, ask students to solve the same problem using the distributive property and an area model.



Note that both methods find the same product of 0.36.

- Provide students with a model of at least one decimal number and an operation and ask them to interpret the model and write a corresponding math sentence. For example, begin with the model shown.

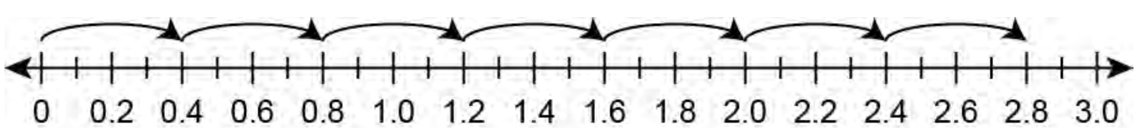


Interpret the model as either $0.23 + 0.23 + 0.23 = 0.69$ or $3 \times 0.23 = 0.69$.

How can a strategy for solving a problem with decimal numbers be written and explained?

M.P.1. Make sense of problems and persevere in solving them. Use properties of operations and place value thinking to simplify a decimal multiplication problem. For example, to multiply 23×0.8, properties of operations can be used to decompose 23 into $(20 + 3)$ and 0.8 into 8×0.1. The resulting expression, $(20 + 3) \times 8 \times 0.1$, can be simplified, using the distributive property, to get $(20 \times 8 + 3 \times 8) \times 0.1$, which is equivalent to $(160 + 24) \times 0.1$ or 184×0.1, which is equal to 18.4. This is a reasonable result since multiplying 23 by a decimal less than 1 will yield a result less than 23. Additionally, use repeated addition and a number line to explain a multiplication problem. Further, use a place value table or expanded form to explain the meaning of sums and differences of decimal numbers.

- Provide students with a multiplication problem involving a whole number and a decimal number. Ask students to find the product using a number line and repeated addition. For example, begin with the problem 0.4×7. Ask students to draw a number line with tenth tick marks. Then, ask students to count off 4 tick marks seven times and note the location of 2.8 on the number line.



- Provide students with two decimal numbers and ask them to write the numbers in expanded form. Then, ask students to find the sum of the two numbers by finding the sum of each place value. For example, begin with the numbers 1.23 and 4.56. The first number can be written as shown.

$$1 \times 1 + 2 \times \frac{1}{10} + 3 \times \frac{1}{100}$$

The second number can be written in the following way.

$$4 \times 1 + 5 \times \frac{1}{10} + 6 \times \frac{1}{100}$$

The two numbers added together can be written as the following.

$$1 \times 1 + 2 \times \frac{1}{10} + 3 \times \frac{1}{100} + 4 \times 1 + 5 \times \frac{1}{10} + 6 \times \frac{1}{100}$$

Then multiplying leads to $1 + \frac{2}{10} + \frac{3}{100} + 4 + \frac{5}{10} + \frac{6}{100}$, which can be combined to get the following results.

$$1 + 4 + \frac{2}{10} + \frac{5}{10} + \frac{3}{100} + \frac{6}{100}$$

$$5 + \frac{7}{10} + \frac{9}{100}$$

This is equivalent to 5.79.

Key Academic Terms:

add, subtract, multiply, divide, decimal, tenths, hundredths, operation, hundred grid

Additional Resources:

- Video: Add & subtract decimals
- Lesson: Adding decimals
- Lesson: Area models for decimal multiplication
- Video: Dividing decimals—modeling

8b

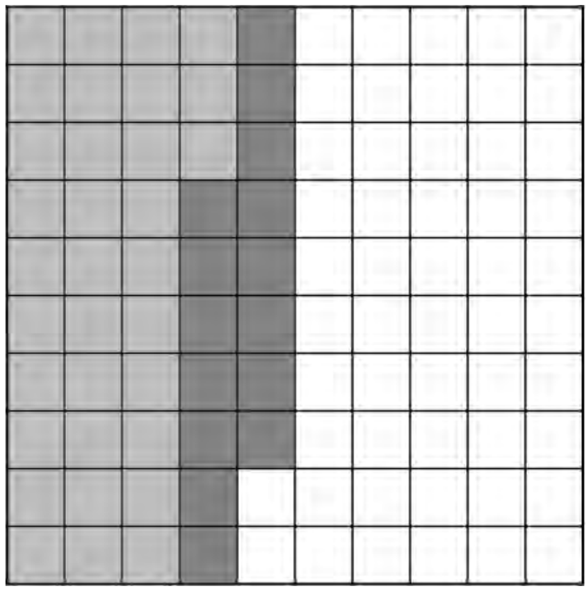
Operations with Numbers: Base Ten
Perform operations with multi-digit whole numbers and decimals to hundredths.
8. Add, subtract, multiply, and divide decimals to hundredths using strategies based on place value, properties of operations, and/or the relationships between addition/subtraction and multiplication/division; relate the strategy to a written method, and explain the reasoning used. b. Solve problems in a real-world context with decimals to hundredths.

Guiding Questions with Connections to Mathematical Practices:

How can real-world problems involving the addition and subtraction of decimals to hundredths be solved using visual models?

M.P.4 Model with mathematics. Identify when real-world problems can be modeled and solved using a decimal grid. For example, a student lives 0.78 miles from school and ran 0.35 miles toward school before walking the remaining distance. The student wants to know how far the remaining distance was. Using a 10 by 10 decimal grid, the problem can be modeled by shading 78 squares (the equivalent of 78 hundredths) and then crossing out 35 of those shaded squares (the equivalent of 35 hundredths) to determine that the remaining 43 squares represent the student walking 0.43 miles to school (the equivalent of 43 hundredths). Additionally, use visual models to solve real-world problems involving the multiplication and division of decimals to hundredths.

- Provide students with a 10 by 10 decimal grid. Ask them to find the sum of two decimals to the hundredths within a real-world context by counting and shading squares. For example, ask students to determine the sum of 0.33 miles and 0.15 miles by first shading 33 squares and then shading an additional 15 squares with a different color.

  Students can then count the total number of shaded squares, 48, to determine that the sum is 0.48 miles.

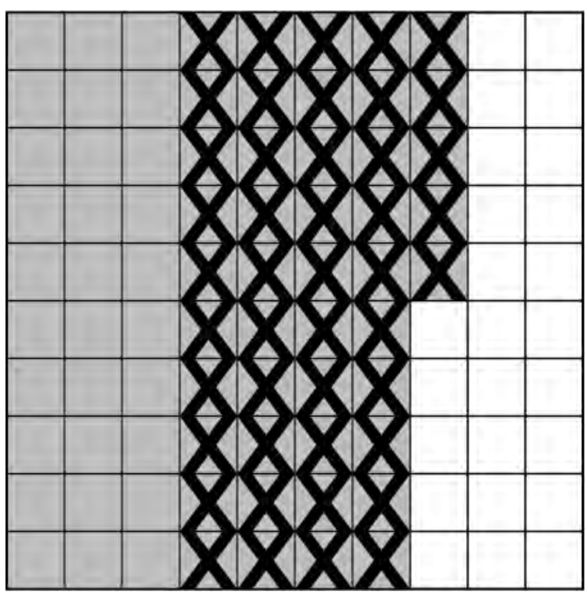
How can real-world problems involving decimals to hundredths be solved using multiple strategies?

M.P.2 Reason abstractly and quantitatively. Identify and implement different strategies for solving real-world problems that involve decimals. For example, a dog owner walks her dog 0.32 miles after breakfast, 0.32 miles after lunch, and 0.32 miles after dinner. The question "How far does the dog owner walk the dog each day?" can be solved using a standard algorithm or using an area model and the distributive property. Additionally, students solve a real-world problem involving decimals to hundredths using one method and then verify the solution using a different method.

- Provide students with a real-world addition or subtraction problem involving decimals to the hundredths. Ask them to first solve it with the standard algorithm and then with a visual model. For example, students can use the standard algorithm to determine the amount of gas remaining after 0.45 gallons is poured from a gas can that originally contained 0.75 gallons.

$$\begin{array}{r} 0.75 \\ -\,0.45 \\ \hline 0.30 \end{array}$$

  The same result of 0.3 gallons can be determined by shading 75 boxes of a 10 by 10 grid and then crossing off 45 of those boxes.



- Provide students with a real-world multiplication problem. Ask them to solve it with the standard algorithm and then check the answer using another method such as a visual model that uses the distributive property. For example, the area of a rectangular key chain that is 3.1 centimeters long and 1.22 centimeters wide can be calculated as 3.782 square centimeters using the algorithm shown.

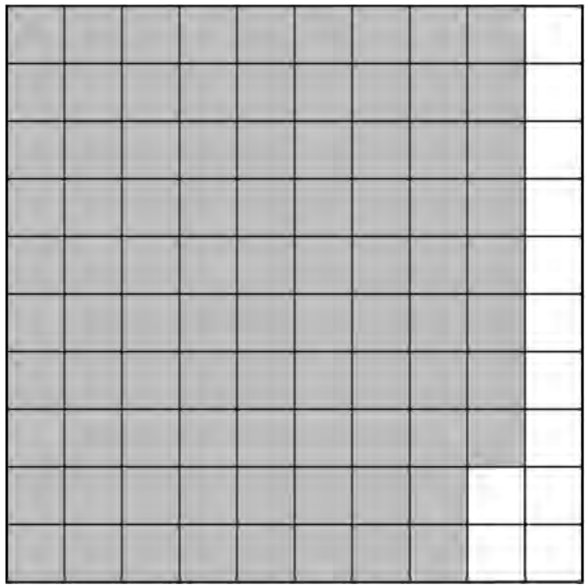
$$\begin{array}{r} 1.22 \\ \times\quad 3.1 \\ \hline 122 \\ +\ 3660 \\ \hline 3.782 \end{array}$$

The product is confirmed using the following model.

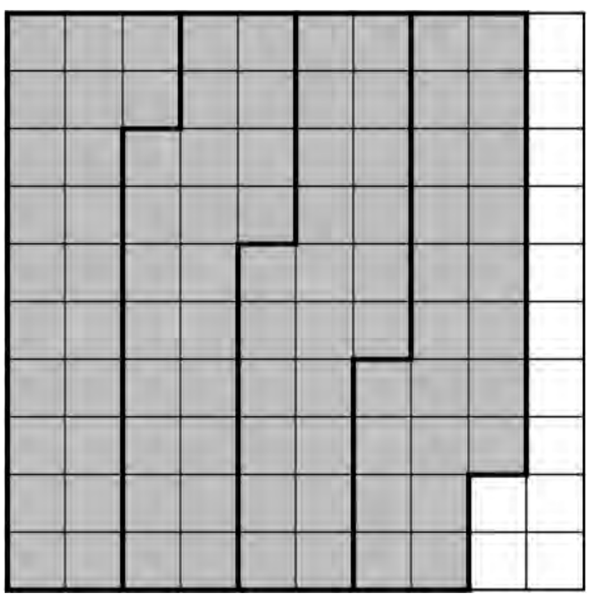
×	1	0.2	0.02
3	3	0.6	0.06
0.1	0.1	0.02	0.002

$$\begin{array}{r} 3.0 \\ 0.6 \\ 0.1 \\ 0.06 \\ 0.02 \\ +\,0.002 \\ \hline 3.782 \end{array}$$

- Provide students with a 10 by 10 decimal grid. Ask them to shade and group squares to represent the quotient of 2 decimals to the hundredths within a real-world context. For example, students could use a 10 by 10 grid to determine how many 0.22-pound hamburger patties can be made from 0.88 pounds of hamburger. In this case, students can first shade in 88 squares on the 10 by 10 grid to represent 0.88.



Next, students can divide the shaded squares into groups of 0.22, which are represented by 22 squares.



As a result, it can now be seen that 4 hamburger patties of 0.22 pounds each can be made from 0.88 pounds of hamburger.

Key Academic Terms:

add, subtract, multiply, divide, decimal, tenths, hundredths, operation, hundred grid

Additional Resources:

- Video: Add & subtract decimals
- Lesson: Adding decimals
- Lesson: Area models for decimal multiplication
- Video: Dividing decimals—modeling

9

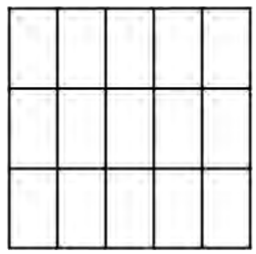
Operations with Numbers: Fractions
Use equivalent fractions as a strategy to add and subtract fractions.
9. Model and solve real-word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally, and assess the reasonableness of answers. *Example*: *Recognize an incorrect result* $\frac{2}{5}+\frac{1}{2}=\frac{3}{7}$ *by observing that* $\frac{3}{7}<\frac{1}{2}$.

Guiding Questions with Connections to Mathematical Practices:

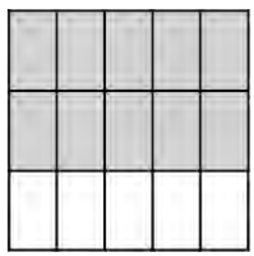
How can word problems involving the addition and subtraction of fractions be represented and solved?

M.P.4. Model with mathematics. Represent word problems that use fractions with visual fraction models, and then use those models to add or subtract. For example, "If Mari and Nina eat $\frac{1}{3}$ and $\frac{2}{5}$ of a pan of brownies, a 3×5 rectangular grid can be used to determine the total amount of the brownies eaten." The portion of the brownies Mari ate can be represented by shading 5 pieces in the grid, and the portion of the brownies Nina ate can be represented by shading 6 pieces in the grid. A total of 11 shaded pieces, from a whole of 15 pieces, models that $\frac{1}{3}+\frac{2}{5}=\frac{11}{15}$. Additionally, represent and solve multistep word problems or word problems that include more than two fractions.

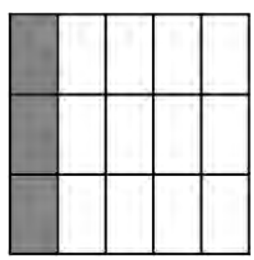
- Ask students to represent a word problem that involves the addition or subtraction of two fractions with a model and determine the answer. For example, "Abby has 1 full can of oil. She pours $\frac{2}{3}$ of the oil in her lawn mower and $\frac{1}{5}$ of the oil in her weed eater. What fraction of the oil remains in the can?" Represent the problem with a rectangle divided into three equal sections with horizontal lines and five equal sections with vertical lines, creating a total of 15 smaller rectangles (fifteenths).



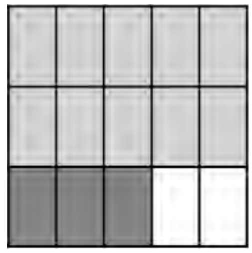
The fraction $\frac{2}{3}$ can be represented by shading 10 fifteenths.



The fraction $\frac{1}{5}$ can be represented by shading 3 fifteenths.



As such, $\frac{13}{15}$ of the oil in the can was used, which means that $\frac{2}{15}$ of the oil remains in the can.



- Ask students to solve word problems that involve adding or subtracting several fractions using models or equations. For example, "Jon, Kasey, Lara, and Molly painted a fence. Jon painted $\frac{1}{3}$ of the fence, Kasey painted $\frac{1}{4}$ of the fence, Lara painted $\frac{3}{10}$ of the fence, and Molly painted the rest of the fence. What fraction of the fence did Molly paint?" First, set up the expression $\frac{1}{3}+\frac{1}{4}+\frac{3}{10}$ to determine what fraction of the fence was painted by Jon, Kasey, and Lara. After finding a common denominator of 60, represent the expression as $\frac{20}{60}+\frac{15}{60}+\frac{18}{60}$, which is equal to $\frac{53}{60}$. Determine the fraction of the fence Molly painted by subtracting $\frac{53}{60}$ from $\frac{60}{60}$. As such, Molly painted $\frac{7}{60}$ of the fence because $\frac{60}{60}-\frac{53}{60}=\frac{7}{60}$.

How can benchmark fractions and number sense be used to estimate and determine whether an answer is reasonable as the sum or difference of two fractions?

M.P.4. Model with mathematics. Compare the relative sizes of fractions by using benchmark fractions. For example, $\frac{4}{7}+\frac{3}{8}$ cannot be equal to $\frac{7}{15}$ because $\frac{7}{15}$ has a value less than $\frac{1}{2}$, and the sum should have a value greater than $\frac{1}{2}$ since $\frac{4}{7}$ is greater than $\frac{1}{2}$. Additionally, use benchmark fractions to determine a reasonable range for the sum of three or more fractions with unlike denominators.

- Ask students to estimate the sums or differences of pairs of fractions to the nearest whole, half, or quarter. For example, ask students to estimate the difference of $2\frac{9}{16}$ and $1\frac{3}{11}$ to the nearest quarter. Observe that $\frac{3}{11}$ is slightly greater than $\frac{1}{4}$ while $\frac{9}{16}$ is slightly greater than $\frac{1}{2}$. As such, the difference between $2\frac{9}{16}$ and $1\frac{3}{11}$ is close to $1\frac{1}{4}$ because 2 – 1 equals 1 while $\frac{1}{2}-\frac{1}{4}$ equals $\frac{1}{4}$.

- Ask students to use benchmark fractions to determine a range for the sum of three fractions with unlike denominators by calculating a minimum and maximum value. For example, give students the fractions $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{8}$. Since all three fractions are greater than $\frac{1}{4}$, it can be concluded that the sum of the fractions must be greater than $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$ or $\frac{3}{4}$. Similarly, by noting that all three fractions are less than $\frac{1}{2}$, conclude that the sum of the fractions must be less than $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$ or $1\frac{1}{2}$. As such, the sum of $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{8}$ must be between $\frac{3}{4}$ and $1\frac{1}{2}$.

Key Academic Terms:

fraction, denominator, numerator, addition, subtraction, visual model, equivalent fractions, equation, mixed number, benchmark fraction

Additional Resources:

- Tutorial: Adding and subtracting fractions with unlike denominators
- Lesson: Adding and subtracting fractions
- Lessons: Addition and subtraction of fractions
- Lesson: Let's have a fraction party!
- Lesson: Fraction World

10

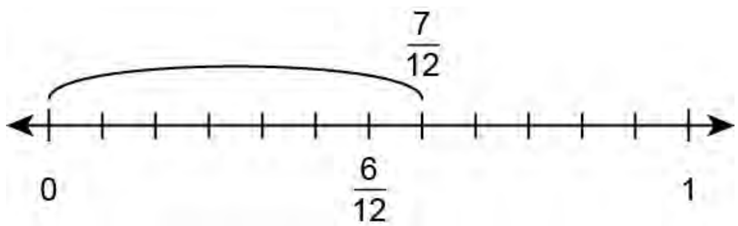
Operations with Numbers: Fractions
Use equivalent fractions as a strategy to add and subtract fractions.
10. Add and subtract fractions and mixed numbers with unlike denominators, using fraction equivalence to calculate a sum or difference of fractions or mixed numbers with like denominators.

Guiding Questions with Connections to Mathematical Practices:

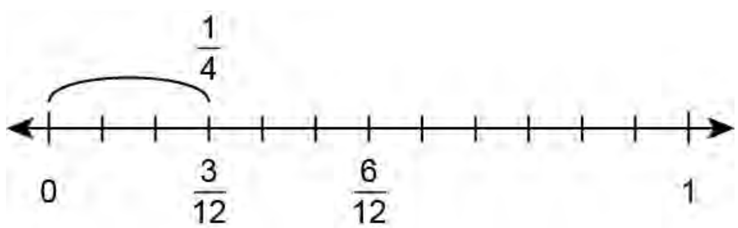
How does adding or subtracting fractions with unlike denominators connect to adding or subtracting fractions with like denominators?

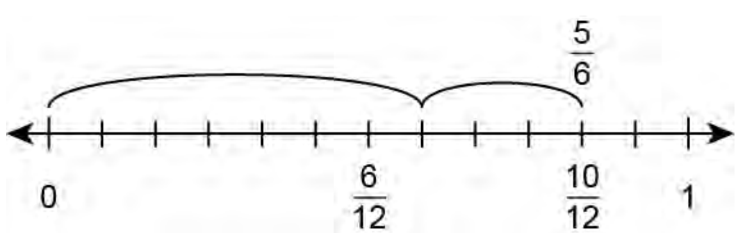
M.P.7. Look for and make use of structure. Connect the ideas that fractions need to be expressed in the same unit fraction to be added and that once equivalent fractions are created with like denominators, the process is the same as traditional addition. For example, adding $\frac{4}{5} + \frac{2}{5}$ is the same as adding the unit fraction $\frac{1}{5}$ six times for a sum of $\frac{6}{5}$. However, $\frac{3}{4}$ and $\frac{3}{5}$ are not expressed in the same unit fraction, so to be added together they must be rewritten so that they are. Writing the sum with a unit fraction of $\frac{1}{20}$ using the equivalent fractions of $\frac{15}{20} + \frac{12}{20}$ makes the solution $\frac{27}{20}$ or $1\frac{7}{20}$. Expressing the fractions with a common denominator ensures like-size pieces are being added. Additionally, use models to represent addition and subtraction of fractions with unlike denominators that are similar to models used to add and subtract fractions with like denominators.

- Ask students to show the sum or difference of two fractions with like denominators when provided with a number sentence. Then, identify an equivalent number sentence that shows the sum or difference of two fractions with unlike denominators. For example, provide the equation $\frac{7}{12}+\frac{3}{12}=\frac{10}{12}$, then write the equivalent equation $\frac{7}{12}+\frac{1}{4}=\frac{5}{6}$. Number lines can also be used to model the addition sentences, as shown.

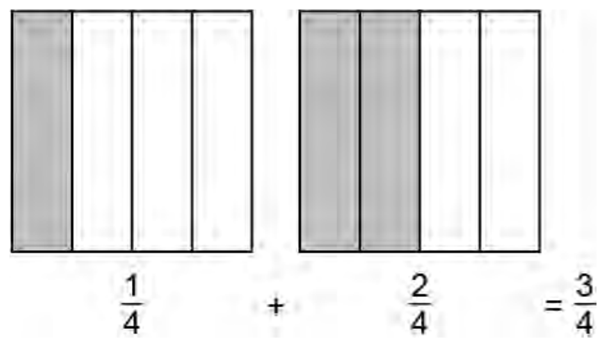


$$\frac{3}{12}=\frac{1}{4}$$

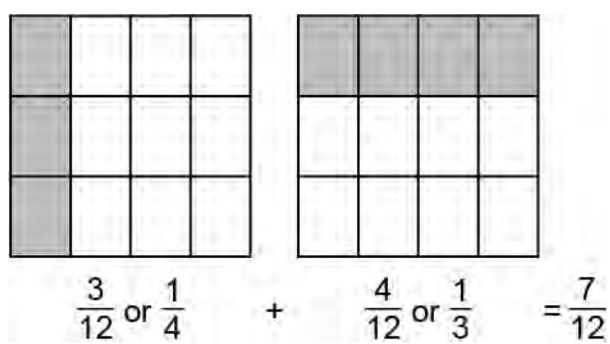




- Ask students to represent and determine the sum of two fractions with like denominators by counting the total number of shaded sections in a rectangular fraction model. Then, ask students to represent and determine the sum of two fractions with unlike denominators by counting the total number of shaded sections in a fraction model that has been manipulated so that both fractions express the same unit. For example, represent and determine the sum of $\frac{1}{4}$ and $\frac{2}{4}$ using a rectangular fraction model.



Notice that 1 shaded one-fourth strip plus 2 shaded one-fourth strips equals a total of 3 shaded one-fourth strips. Next, ask students to represent and determine the sum of $\frac{1}{4}$ and $\frac{1}{3}$ using rectangular fraction models. After partitioning both rectangles in one-third and one-fourth strips, count a total of 7 shaded squares, or $\frac{7}{12}$ of the rectangle.

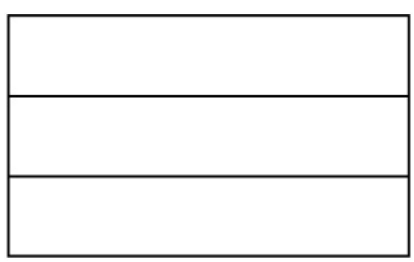


How can the common denominator between fractions with unlike denominators be determined in order to add or subtract?

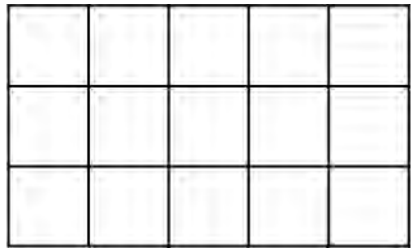
M.P.8. Look for and express regularity in repeated reasoning. Solve a variety of problems to find the general method that $\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$ or $\frac{a}{b} - \frac{c}{d} = \frac{(ad - bc)}{bd}$. For example, after solving a variety of fraction subtraction problems with many different denominators, explore the similarities when solving to find that a like denominator can always be found by multiplying each numerator and denominator by the other fraction's denominator to give an equivalent fraction, making it possible to subtract the fractions. Additionally, partition a rectangle horizontally and vertically to represent a common unit of two fractions with unlike denominators.

- Ask students to complete a partially solved problem involving the sum or difference of two fractions with unlike denominators. For example, given the problem $\frac{4}{9} - \frac{2}{7} = \frac{28 - 18}{?} = \frac{10}{?}$, determine that each question mark represents 63.

- Ask students to use rectangular models to solve equations with two fractions that have unlike denominators. Partition a rectangle with horizontal lines to represent the unit of the first fraction and partition the same rectangle with vertical lines to represent the unit of the second fraction. Then, count the total number of smaller rectangles created by the horizontal and vertical lines, noting that it represents a common denominator. For example, begin with the fractions $\frac{1}{3}$ and $\frac{2}{5}$. Partition a rectangle into thirds using horizontal lines.



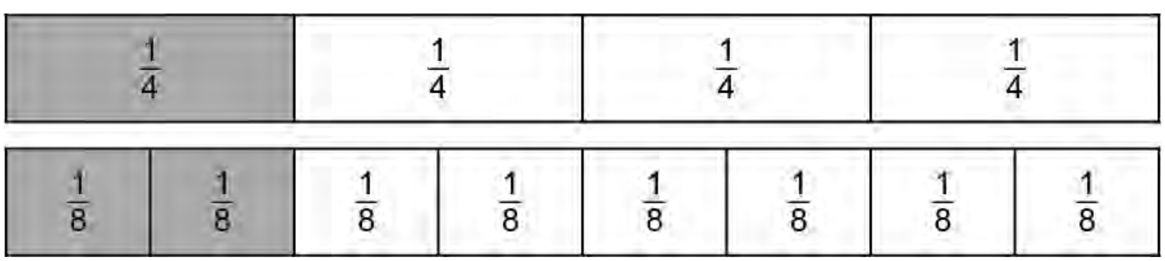
Then, partition the rectangle into fifths using vertical lines.



Note that the total number of small rectangles, 15, can be used to represent a common denominator for the fractions $\frac{1}{3}$ and $\frac{2}{5}$.

M.P.5. Use appropriate tools strategically. Use a variety of strategies, including visual models and equations, to add fractions with unlike denominators by finding equivalent fractions. For example, add $1\frac{2}{3}+\frac{7}{8}$ by multiplying the denominators 3 and 8 to find a common denominator of 24, then solve $1+\frac{2}{3}\times\frac{8}{8}+\frac{7}{8}\times\frac{3}{3}=1+\frac{16}{24}+\frac{21}{24}=1+\frac{37}{24}=2\frac{13}{24}$. Additionally, use fraction strips as a tool to generate equivalent fractions.

- Ask students to solve an addition or subtraction problem that includes fractions with unlike denominators. Use fraction strips to change one or both fractions in order to share a common denominator. Then, complete the problem. For example, begin with $\frac{5}{8}-\frac{1}{4}$. Using fraction strips, model that $\frac{1}{4}$ is equivalent to $\frac{2}{8}$.



As such, the problem can be written as $\frac{5}{8}-\frac{2}{8}$, which is equivalent to $\frac{3}{8}$.

- Give students an addition or subtraction problem that includes fractions with unlike denominators, and ask students to make a partial list of the multiples of each denominator. Then, have students write equivalent fractions using a common multiple for the denominators and complete the problem. For example, give students the problem $\frac{3}{4}+\frac{1}{6}$. List the multiples of 4 as 4, 8, 12, 16, 20 and the multiples of 6 as 6, 12, 18, 24, 30. Then, identify a common multiple as 12. Rewrite both fractions using 12 as the denominator and solve $\frac{9}{12}+\frac{2}{12}=\frac{11}{12}$.

Key Academic Terms:

fraction, denominator, numerator, addition, subtraction, visual model, equivalent fractions, equation, mixed number

Additional Resources:

- Tutorial: Adding and subtracting fractions with unlike denominators
- Lesson: Adding and subtracting fractions
- Lessons: Addition and subtraction of fractions
- Lesson: Let's have a fraction party!
- Lesson: Fraction World

11a

Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
11. Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers. a. Model and interpret a fraction as division of the numerator by the denominator ($\frac{a}{b} = a \div b$).

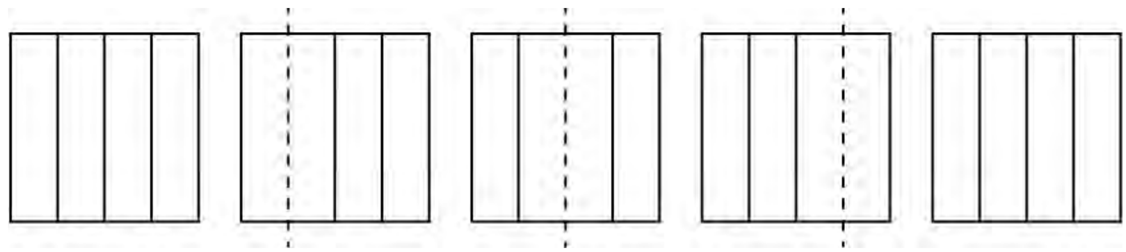
Guiding Questions with Connections to Mathematical Practices:

How can a fraction be interpreted as a division problem?

M.P.8. Look for and express regularity in repeated reasoning. Know that a fraction represents the quotient of two quantities: namely, the numerator and the denominator. For example, the quotient of 2 and 5 can be represented with either the expression $2 \div 5$ or the fraction $\frac{2}{5}$. Additionally, use diagrams or pictures to represent fractions as division problems.

- Ask students to write corresponding expressions of the form $a \div b$ when given a fraction. For example, given the fractions $\frac{7}{3}$, $\frac{11}{9}$, and $\frac{1}{13}$, write the expressions $7 \div 3$, $11 \div 9$, and $1 \div 13$. Alternatively, provide students with an expression of the form $a \div b$ and ask them to write a corresponding fraction. For example, given the expression $17 \div 5$, write the fraction $\frac{17}{5}$.

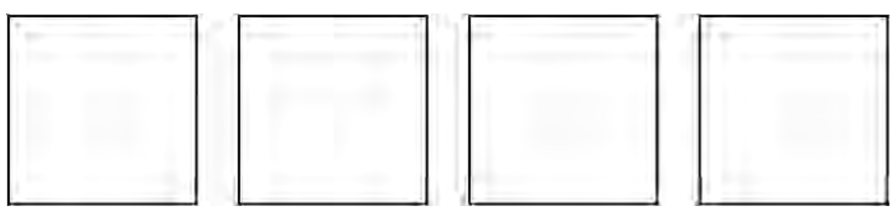
- Ask students to identity fractions and write corresponding expressions of the form $a \div b$ when given a picture or diagram that represents a fraction. For example, provide students with the following diagram:



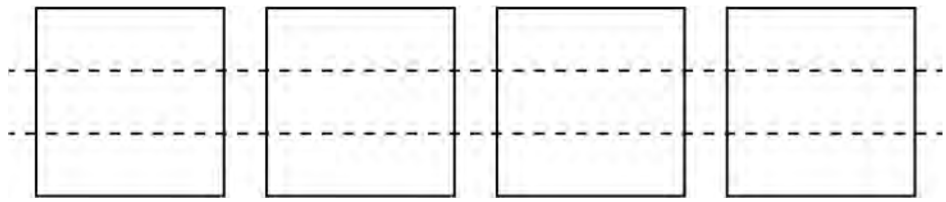
Students should study the diagram to see that there are 5 wholes (squares) that are divided equally into 4 parts. As such, the diagram represents the fraction $\frac{5}{4}$ and can be written as $5 \div 4$.

M.P.7. Look for and make use of structure. Use multiplicative thinking to decompose and understand fractions as division. For example, interpret $\frac{7}{6}$ as 7 divided by 6 and also as the product of $\frac{1}{6}$ and 7. Additionally, use visuals to illustrate multiplicative thinking or repeated addition thinking in order to understand fractions as division.

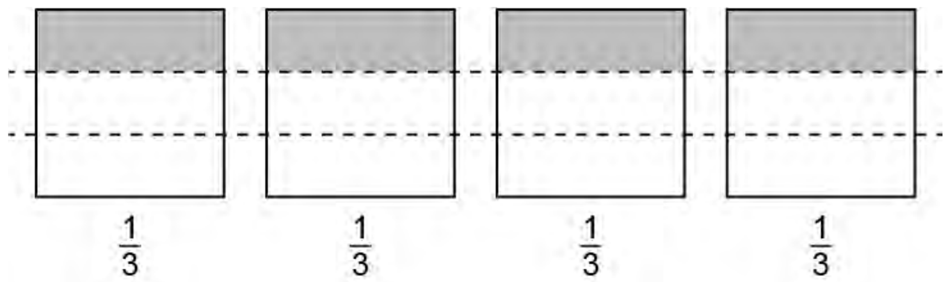
- Ask students to represent the numerators of fractions as "whole" squares given a picture or diagram. Divide each "whole" into a number of equally sized sections equivalent to the denominator; then add the fractional equivalent of 1 section from each of the wholes and compare the sum to the original fraction. For example, begin with the fraction $\frac{4}{3}$ and draw 4 squares, or "wholes."



Then, divide each square, or "whole," into 3 equally sized sections, corresponding to the denominator.



Next, write the fractional equivalent of one section of each "whole" beneath each square.



Finally, add all the fractions together and note that the sum is equivalent to the original fraction, $\frac{4}{3}$. As such, the fraction $\frac{4}{3}$ can be thought of as $\frac{1}{3}$ taken 4 times or as $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$.

- Ask students to write an equivalent multiplication expression, an equivalent fraction, and an equivalent division expression showing the sum of unit fractions. For example, given the expression $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$, write $7 \times \frac{1}{5}$, $\frac{7}{5}$, and $7 \div 5$.

Key Academic Terms:

fraction, numerator, denominator, division, remainder, dividend, divisor

Additional Resources:

- Lesson: Fractions as division

  Converting fractions of a unit into a smaller unit

11b

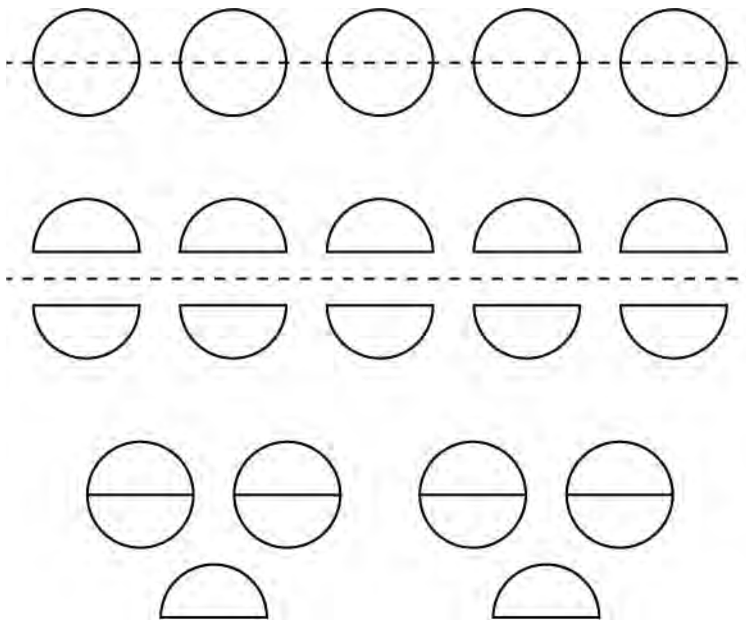
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
11. Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers. b. Use visual fraction models, drawings, or equations to represent word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers.

Guiding Questions with Connections to Mathematical Practices:

How can word problems involving the division of whole numbers be represented visually?

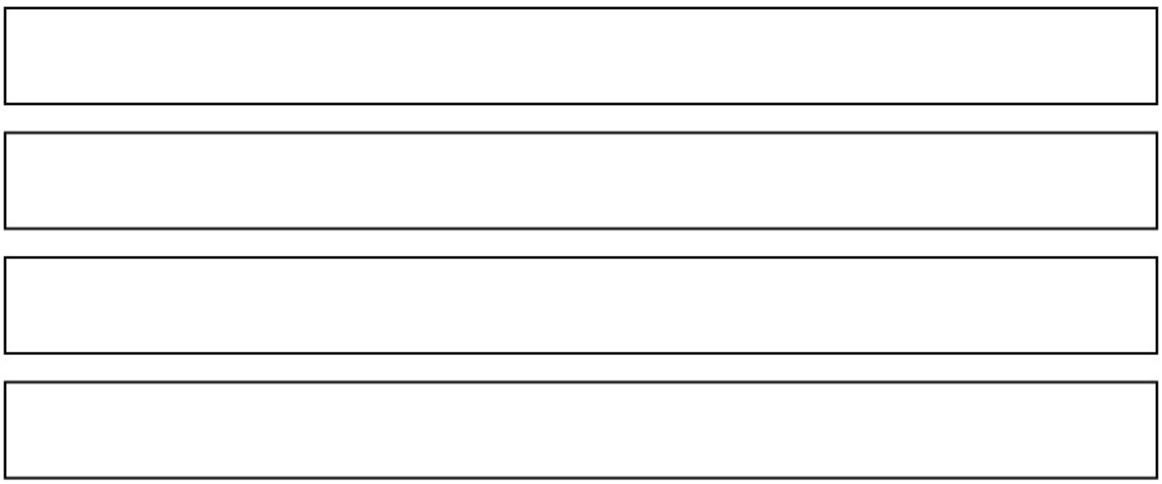
M.P.4. Model with mathematics. Model the quotient of two whole numbers by starting with a whole number of objects equal to the dividend (numerator), and then partitioning those objects equally into a number of sections that is equivalent to the divisor (denominator). For example, if 2 pies are divided equally among 10 people, then an image of 2 whole pies that are segmented into 10 equal sections illustrates that each person receives $\frac{2}{10}$ of a pie. Additionally, use tape diagrams to model quotients of whole numbers.

- Ask students to interpret a model and create a corresponding word problem for a visual model that shows the division of two whole numbers. For example, begin with the following model.

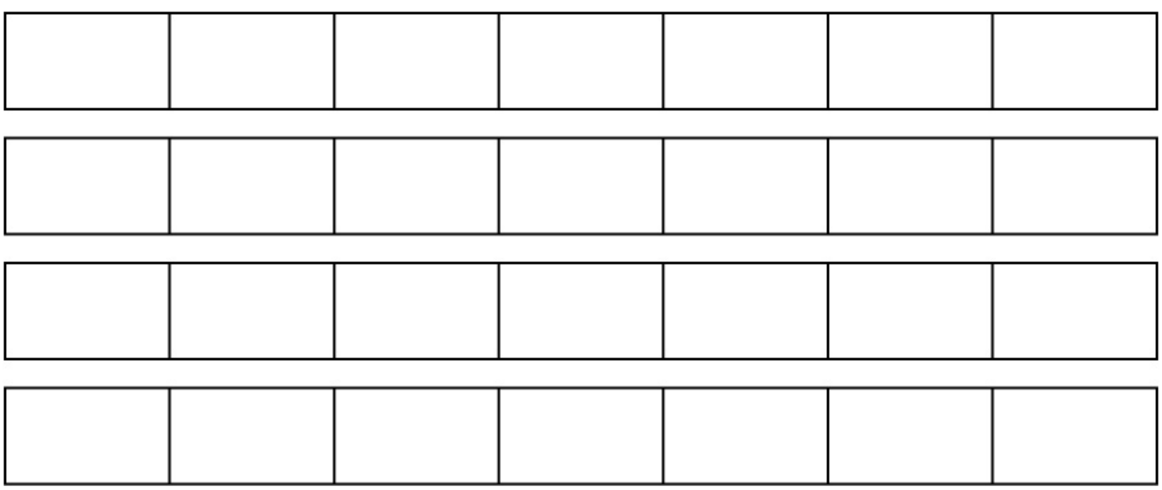


After observing that 5 wholes are divided in half, write a problem such as one in which 5 cookies are divided equally between two friends so that each friend receives $2\frac{1}{2}$ cookies. Note that this model can also be interpreted as 5 wholes divided into pieces of size $2\frac{1}{2}$, instead of 5 divided by 2, and that would also be a correct interpretation.

- Ask students to solve word problems involving the division of whole numbers that can be solved using a tape diagram and ask students to verify answers using repeated addition or multiplication. For example, give students the problem "4 cans of fruit juice are shared equally among 7 teammates. What fraction of a can of fruit juice does each teammate receive? Represent the problem with a tape diagram in which each bar represents 1 can of fruit juice."



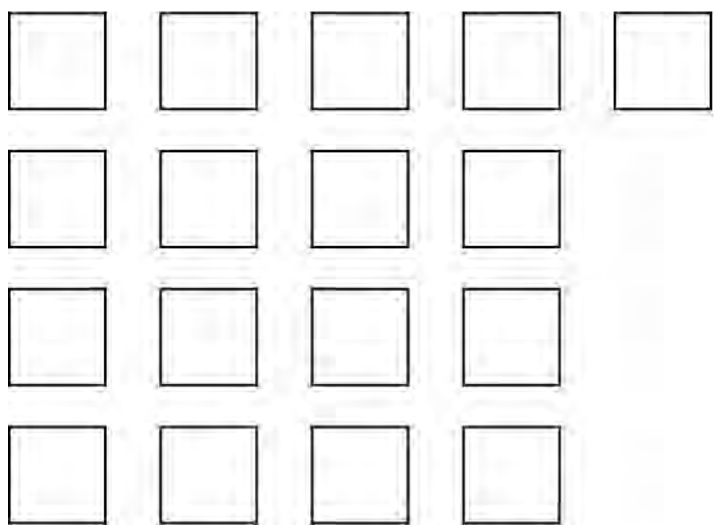
Dividing each bar into 7 equal sections can be used to show the fraction of a can of fruit juice that each teammate receives.



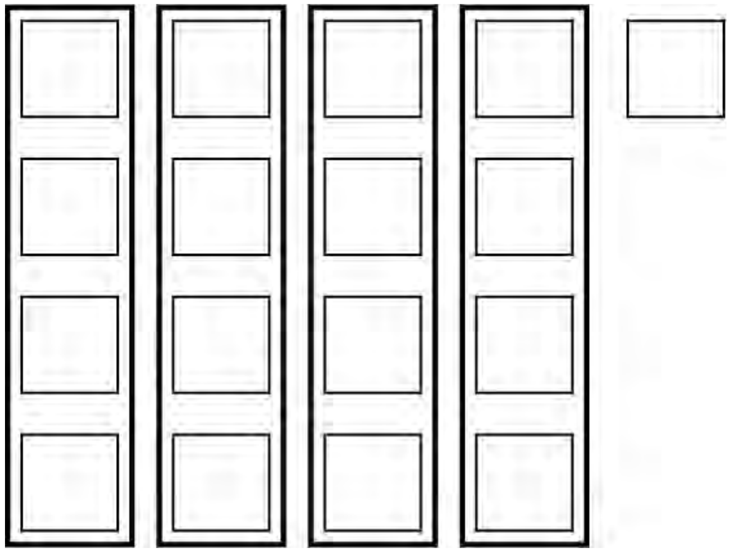
If each teammate receives 1 section from each can, then each teammate receives a total serving size of $4 \div 7$ or $\frac{4}{7}$ cans. Verify the answer by adding $\frac{4}{7}+\frac{4}{7}+\frac{4}{7}+\frac{4}{7}+\frac{4}{7}+\frac{4}{7}+\frac{4}{7}$ for a total of $\frac{28}{7}$, which is the same as 4.

M.P.2. Reason abstractly and quantitatively. Represent quotients as fractions greater than 1 or whole numbers with fractional remainders, depending on the context. For example, given that a bakery divides 170 cups of flour equally among 20 bowls, find how many cups of flour are in each bowl by writing the expression $\frac{170}{20}$ and solve to find a solution of $\frac{17}{2}$ or $8\frac{1}{2}$ cups, and use $8\frac{1}{2}$ cups to represent the solution given the context. Additionally, draw pictures that illustrate quotients of whole numbers that have fractional remainders.

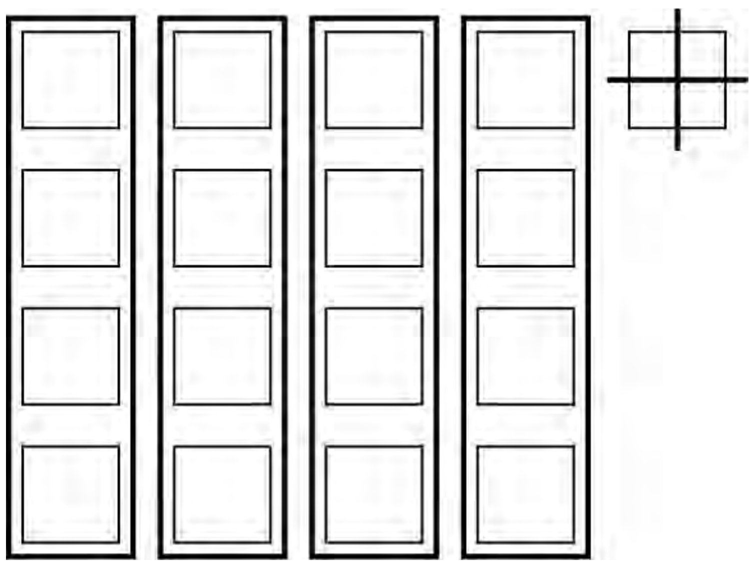
- Ask students to solve word problems with an answer that can be expressed as a quotient greater than 1 by drawing a picture and expressing the answer as a fraction. For example, "A gardener has 17 quarts of potting soil. He divides the potting soil equally into 4 pots. How much potting soil does he put into each pot?" Represent 17 quarts with 17 squares.



Then, divide as many complete squares as possible into 4 equal groups.

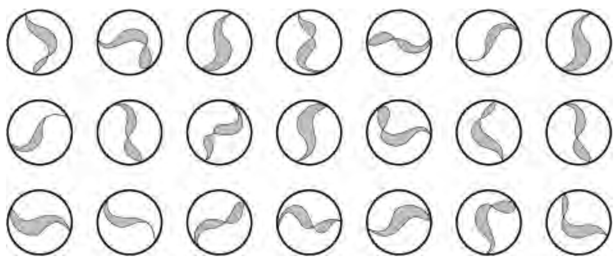


Finally, divide the remaining square into 4 equal parts.

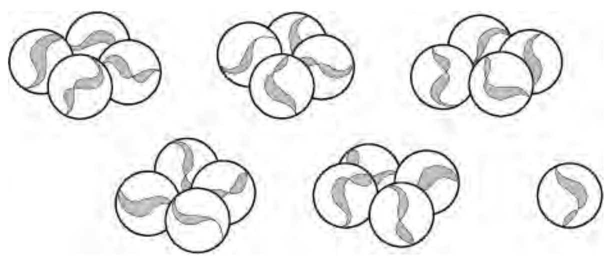


Note that each of the 4 pots receives $4\frac{1}{4}$ quarts of potting soil. As such, $\frac{17}{4}$ is equal to $4\frac{1}{4}$.

- Ask students to solve problems involving the quotient of two whole numbers and write corresponding fractions with explanations for remainders. For example, given the problem "Josie has 21 marbles that she wants to divide equally among her 5 siblings. How many marbles does each sibling receive?" students should represent the problem with a fraction. Then, draw a picture to illustrate the number of marbles that each sibling gets and explain the meaning of the remainder. In this case, Josie wants to divide 21 marbles into 5 groups, which is equivalent to $\frac{21}{5}$. Draw an array to show 21 marbles.



Next, divide the marbles into 5 equal groups of 4 marbles.



In this context, there is one remaining marble to be divided among 5 siblings, which is $\frac{1}{5}$. Since marbles cannot be divided, rather than each sibling receiving $4\frac{1}{5}$ marbles, each sibling receives 4 marbles, with 1 marble left over.

Key Academic Terms:

fraction, numerator, denominator, division, remainder, dividend, divisor

Additional Resources:

- Lesson: Fractions as division
- Activity: Converting fractions of a unit into a smaller unit

12a

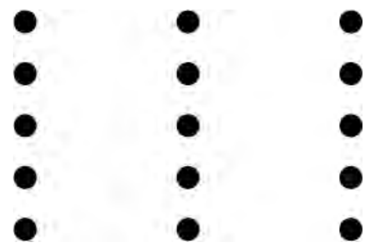
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
12. Apply and extend previous understandings of multiplication to find the product of a fraction times a whole number or a fraction times a fraction. a. Use a visual fraction model (area model, set model, or linear model) to show $(\frac{a}{b}) \times q$ and create a story context for this equation to interpret the product as *a* parts of a partition of *q* into *b* equal parts.

Guiding Questions with Connections to Mathematical Practices:

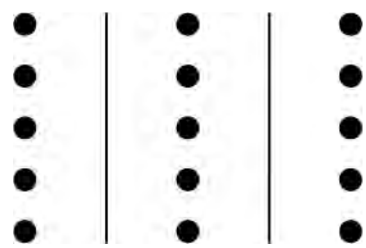
How can knowledge of multiplication and division be used to represent fraction multiplication in a variety of ways?

M.P.7. Look for and make use of structure. Use the meanings of multiplication and division, the relationship between multiplication and division, and visual models to solve products of fractions. For example, use a visual fraction model to show that since $\frac{1}{5} \times 2$ is 1 part of a partition of 2 into 5 equal parts and is equal to $\frac{2}{5}$, then $\frac{3}{5} \times 2$ is 3 parts of a partition of 2 into 5 equal parts, which is equal to $\frac{6}{5}$. Additionally, illustrate fraction multiplication by creating visual models.

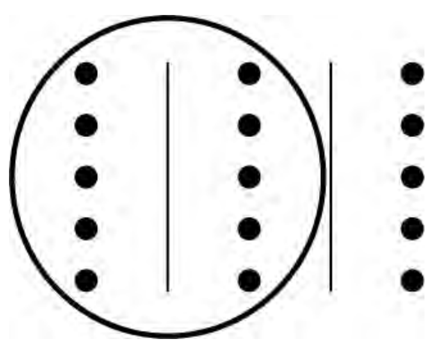
- Ask students to create a visual model with a problem of the form $(\frac{a}{b}) \times q$ and then create a story context that fits the model. First, represent q with an array. Then, partition the array into b equal parts. Finally, circle and count the number of dots in a parts of the partition. For example, begin with $(\frac{2}{3}) \times 15$. Then, draw an array that represents 15.



Then, partition the array into 3 equal parts.



Finally, circle and count the number of dots in 2 of the parts.



Emphasize and discuss how the model illustrates that $\frac{2}{3}$ of 15 is 10. A possible context is "Otto needs \$15 to buy a book. He has $\frac{2}{3}$ of the money needed. How much money does he have?" The \$15 is split into 3 equal parts of \$5 each. Otto has 2 of the 3 parts, so he has \$10.

- Ask students to write an equivalent expression to a fraction problem of the form $(\frac{a}{b}) \times q$ that does not contain a fraction. For example, provide students the expression $(\frac{2}{7}) \times 14$. Then, students write the equivalent expression $2 \times 14 \div 7$.

Key Academic Terms:

fraction, fraction model, whole number, multiplication, division, numerator, denominator, product, equation, expression

Additional Resources:

- Activity: Fraction scoot game
- Activity: Connecting the area model to context

12b

Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
12. Apply and extend previous understandings of multiplication to find the product of a fraction times a whole number or a fraction times a fraction. b. Use a visual fraction model (area model, set model, or linear model) to show $(\frac{a}{b}) \times (\frac{c}{d})$ and create a story context for this equation to interpret the product.

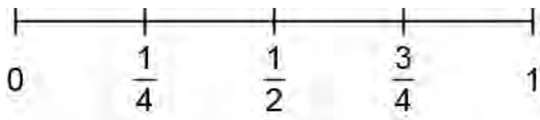
Guiding Questions with Connections to Mathematical Practices:

How is multiplying a whole number by a fraction similar to multiplying a fraction by a fraction?

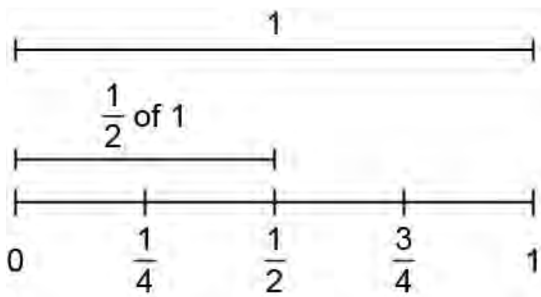
M.P.1. Make sense of problems and persevere in solving them. Observe that any whole number can be written as a fraction with a denominator of 1, and therefore the product of a whole number and a fraction can be written as the product of two fractions. For example, the expression $2 \times \frac{2}{3}$ can be written as $\frac{2}{1} \times \frac{2}{3}$. Additionally, use number lines to visualize the similarities between multiplying a fraction by a whole number and a fraction by a fraction.

- Ask students to write the product of two whole numbers or a product of a whole number and a fraction and then write an equivalent expression as a product of two fractions. For example, when given the expressions 2×6 and $4 \times \frac{7}{9}$, write the expressions $\frac{2}{1} \times \frac{6}{1}$ and $\frac{4}{1} \times \frac{7}{9}$.

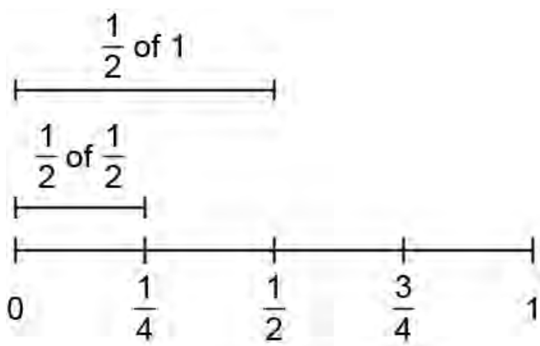
- Ask students to show the product of a fraction and a whole number with a number line and an expression. Use the number line to visualize the product. Then, provide students with a second expression showing the product of two fractions. Use the number line to visualize the product and note the similarities between the first expression and the second expression. For example, give students the expression $\frac{1}{2} \times 1$ and the following number line.



Locate the whole number 1 and then locate $\frac{1}{2}$ of 1.



Next, provide students with the expression $\frac{1}{2} \times \frac{1}{2}$ and the same number line. Ask students to locate $\frac{1}{2}$, and then ask them to locate $\frac{1}{2}$ of $\frac{1}{2}$.



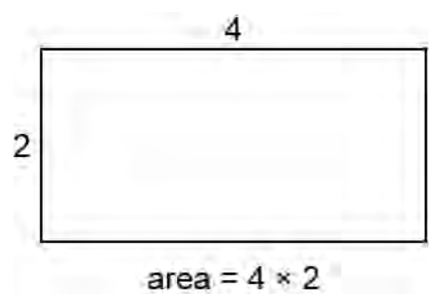
Emphasize that $\frac{1}{2}$ of a fraction can be determined in the same way as $\frac{1}{2}$ of a whole.

What is the general rule for multiplying fractions and how is it shown using a visual model?

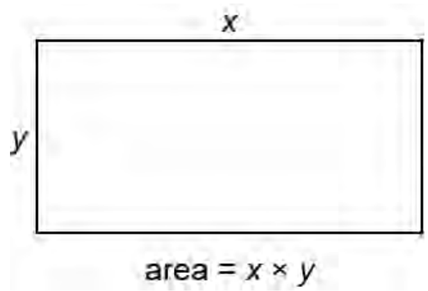
M.P.8. Look for and express regularity in repeated reasoning. After solving fraction multiplication problems in a variety of ways, generalize that when two fractions are multiplied together, the numerator of the product is equal to the product of the two numerators in the fractions and the denominator of the product is equal to the product of the two denominators in the fractions. For example, connect a visual model of the product of $\frac{3}{5}$ and $\frac{4}{7}$ to the expression $\frac{3 \times 4}{5 \times 7}$. Additionally, observe that the general rule for multiplying fractions is related to the procedure for determining the area of a rectangle.

- Ask students to solve real-world problems involving the product of two fractions using the general rule. For example, "Joe spent $\frac{4}{5}$ of an hour doing homework. He spent $\frac{1}{3}$ of that time doing science homework. What fraction of an hour did Joe spend doing science homework?" Determine that Joe spent $\frac{4}{15}$ of an hour doing science homework by multiplying $\frac{4}{5}$ and $\frac{1}{3}$ as $\frac{4 \times 1}{5 \times 3}$.

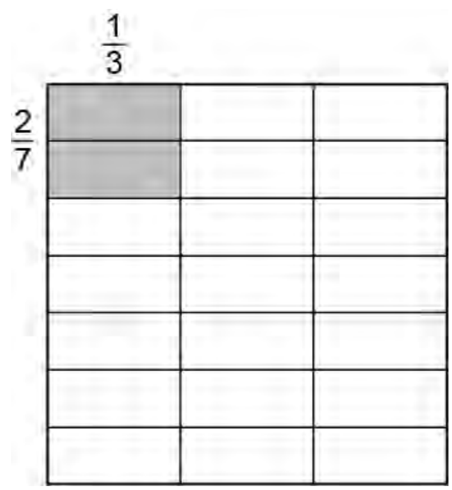
- Ask students to explain the procedure for determining the area of a rectangle. After reviewing that the area of a rectangle is equal to the product of the length and width, ask students to draw and determine the area of a rectangle with a length of 4 units and a width of 2 units.



Next, point out that the product of any two positive numbers, x and y, can be thought of as the area of a rectangle with a length of x units and a width of y units.



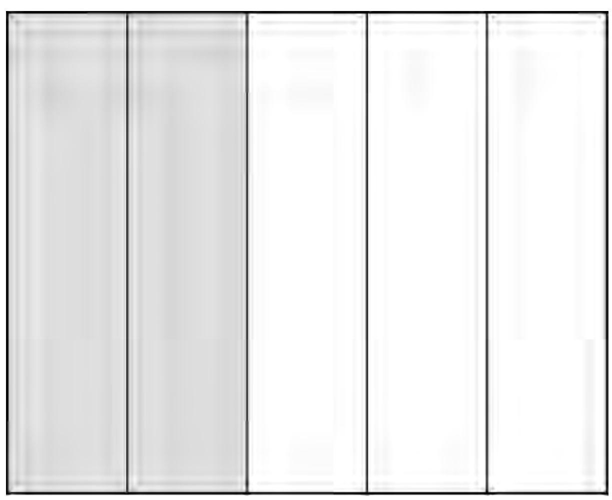
Now, demonstrate the product of two fractions, $\frac{1}{3}$ and $\frac{2}{7}$. Draw a rectangle with a length of $\frac{1}{3}$ and a width of $\frac{2}{7}$ within a square that has side lengths of 1. More specifically, draw a square and divide it vertically into 3 equal sections. Then, divide it again horizontally into 7 equal sections.



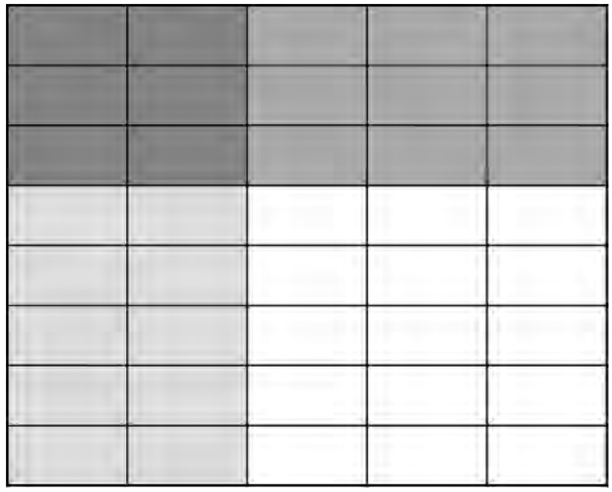
Note that there is now a rectangle (shaded) with a length of $\frac{1}{3}$ and a width of $\frac{2}{7}$ and that its area is equal to $\frac{2}{21}$ of the total square. Emphasize that the product of the numerators is equal to the number of parts represented by the shaded rectangles, while the product of the denominators is equal to the total number of rectangles within the square.

M.P.4. Model with mathematics. Use fraction models to visually represent products that include fractions. For example, if a family had 2 pies and ate $\frac{2}{3}$ of each pie, then the total amount of pie eaten can be represented with 2 circles divided into thirds with 2 segments shaded in each circle. The total of 4 shaded segments shows that the family ate a total of $\frac{4}{3}$ pies (i.e., $2 \times \frac{2}{3} = \frac{4}{3}$). Additionally, interpret the meaning of a fraction model by representing it with a numerical expression.

- Ask students to solve a problem involving the product of two fractions. Students solve the problem using a model and then check the answer using the general rule $(\frac{a}{b}) \times (\frac{c}{d}) = (\frac{ac}{bd})$. For example, "On Monday, Teresa painted $\frac{2}{5}$ of a fence. On Tuesday, she painted a second coat on $\frac{3}{8}$ of what she painted on Monday. What fraction of the fence had two coats of paint?" Represent the work done on Monday with a rectangle divided vertically into 5 equal strips with 2 of the strips shaded.

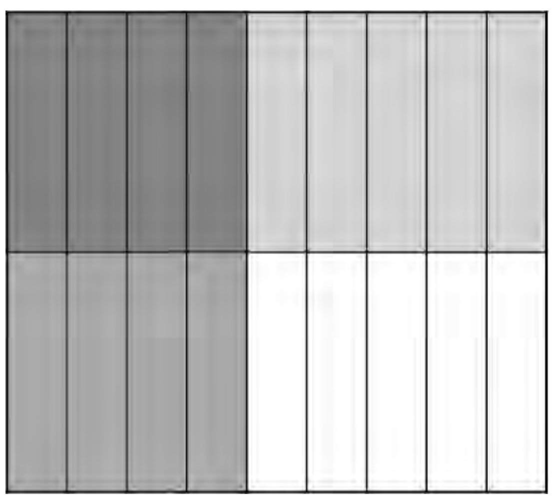


Then, divide the rectangle horizontally into 8 equal strips with 3 of those strips shaded.



The fraction $\frac{6}{40}$ represents the fraction of the fence with two coats of paint. The same answer can be determined using the general rule such that $\frac{2}{5} \times \frac{3}{8} = \frac{6}{40}$.

- Ask students to use a model that represents the product of two fractions to write a corresponding multiplication equation. For example, present students with the following model.



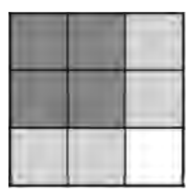
Interpret the model as $\frac{4}{9} \times \frac{1}{2}$ or $\frac{1}{2} \times \frac{4}{9} = \frac{4}{18}$ or $\frac{2}{9}$.

How does context help interpret products of fractions?

M.P.2. Reason abstractly and quantitatively. Write a story problem for an equation to make sense of the product and attend to the whole. For example, for the expression $\frac{5}{8} \times \frac{1}{4}$, write the story problem "There was $\frac{5}{8}$ of a cake left over from a party. Rayshawn ate $\frac{1}{4}$ of the leftover cake. How much of the whole cake did Rayshawn eat?" and solve to get $\frac{5}{32}$ of the whole cake. Additionally, create a context and equation for a fraction model that represents the product of two fractions.

- Ask students to create and solve a contextual problem that involves the product of two fractions and a unit of measurement. For example, begin with the fractions $\frac{7}{8}$ and $\frac{1}{2}$ and the unit "liters." Then, create and solve a problem with the information. For example, "Laurel has $\frac{7}{8}$ liter of salad dressing and then uses $\frac{1}{2}$ of it for a recipe. As a result, she has used $\frac{7}{16}$ liter of salad dressing because $\frac{7}{8} \times \frac{1}{2} = \frac{7}{16}$."

- Ask students to use a fraction model that represents the product of two fractions in order to create a corresponding equation and context. For example, begin with the following model.



  After recognizing that the model corresponds to the product of $\frac{2}{3}$ and $\frac{2}{3}$, create a context in which $\frac{2}{3}$ of the dogs in a dog park are terriers and $\frac{2}{3}$ of the terriers are males. As such, $\frac{4}{9}$ of all the dogs in the dog park are male terriers because $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, just as 4 out of the 9 squares are shaded twice.

Key Academic Terms:

fraction, fraction model, whole number, multiplication, division, numerator, denominator, product, equation, expression

Additional Resources:

- Activity: Fraction scoot game
- Activity: Connecting the area model to context

12c

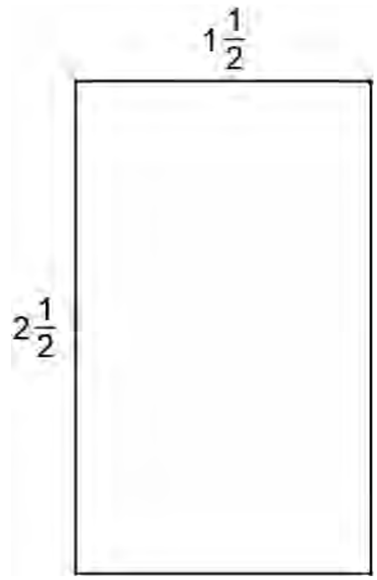
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
12. Apply and extend previous understandings of multiplication to find the product of a fraction times a whole number or a fraction times a fraction. c. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Guiding Questions with Connections to Mathematical Practices:

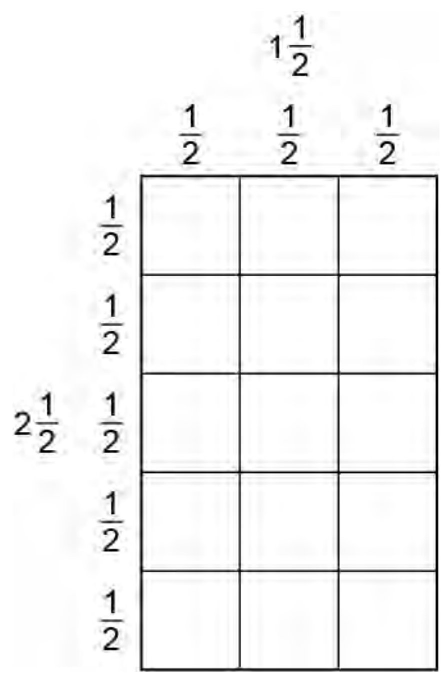
How can tiling with unit squares that have unit fraction side lengths be used to find the area of rectangles with fractional side lengths?

M.P.4. Model with mathematics. Model the area of a rectangle with fractional side lengths by creating a rectangular grid that uses unit fraction side lengths for each dimension. For example, the area of a rectangular poster with a length of $\frac{3}{4}$ meter and a width of $\frac{1}{4}$ meter can be represented by a rectangular grid with 4 columns and 4 rows. When 3 of the 4 columns are shaded with one color to represent $\frac{3}{4}$ and 1 of the 4 rows is shaded with another color to represent $\frac{1}{4}$, then 3 squares from a total of 16 squares are shaded with both colors. The total area of the poster is $\frac{3}{16}$ of a square meter. Additionally, use a ruler, paper, and pencil as tools to draw tiles and determine the areas of rectangles.

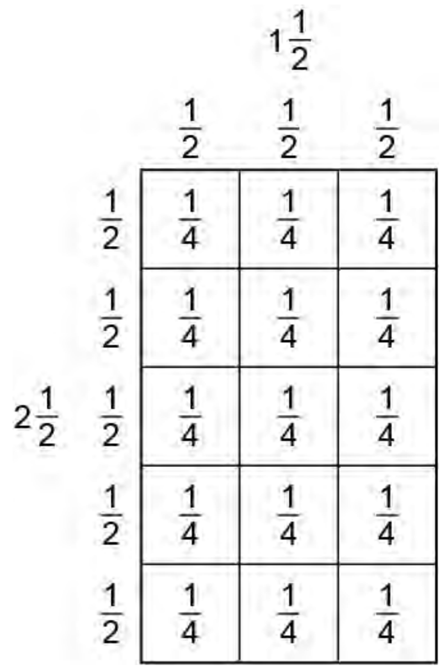
- Ask students to solve real-world or mathematical problems involving determining the area of a rectangle with fractional side lengths by sketching a diagram and tiling it. For example, a rectangular painting canvas has side lengths of $1\frac{1}{2}$ feet and $2\frac{1}{2}$ feet. Determine the area of the canvas by sketching a rectangle and labeling the sides as "$1\frac{1}{2}$" and "$2\frac{1}{2}$."



Now, tile the rectangle using squares that have unit fraction side lengths. In this case, the rectangle can be tiled with squares that have side lengths of $\frac{1}{2}$ foot.



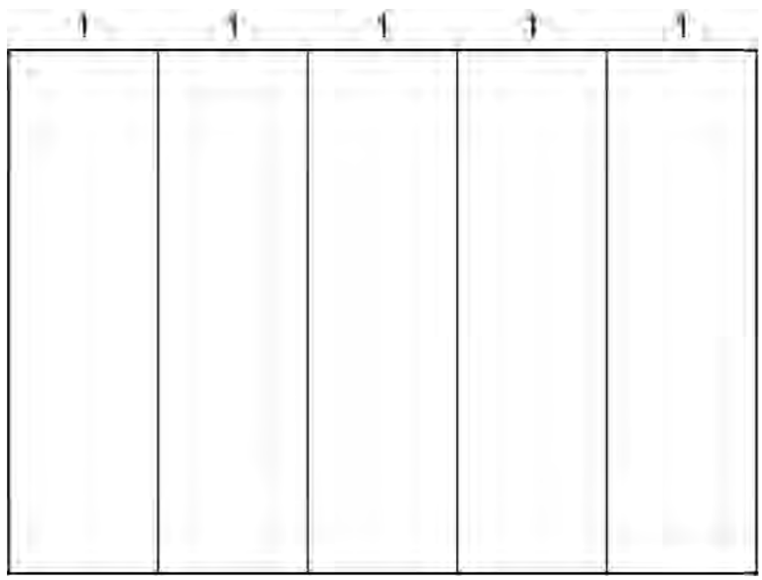
Note that the area of each square is $\frac{1}{4}$ square foot because 4 squares make a unit square.



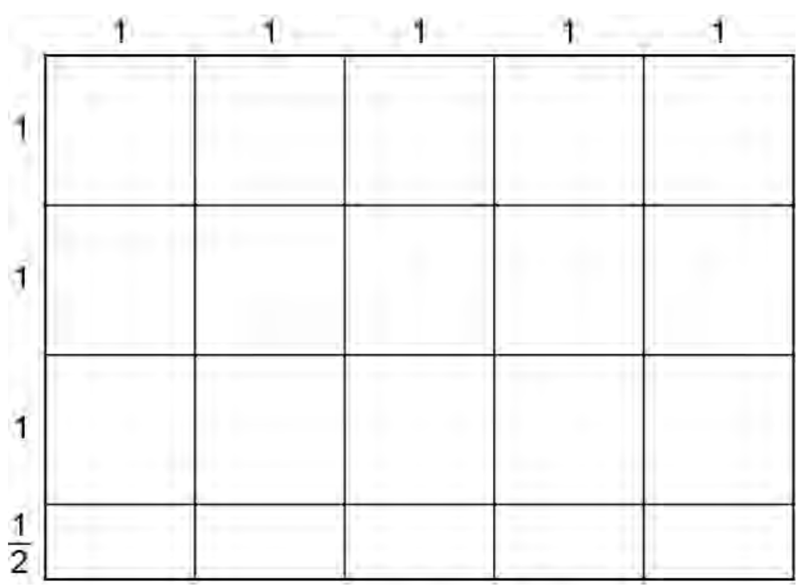
Combine the areas of all the unit squares to determine the total area of the rectangle.

Conclude that the area is $3\frac{3}{4}$ square feet because $15 \times \frac{1}{4} = \frac{15}{4} = 3\frac{3}{4}$.

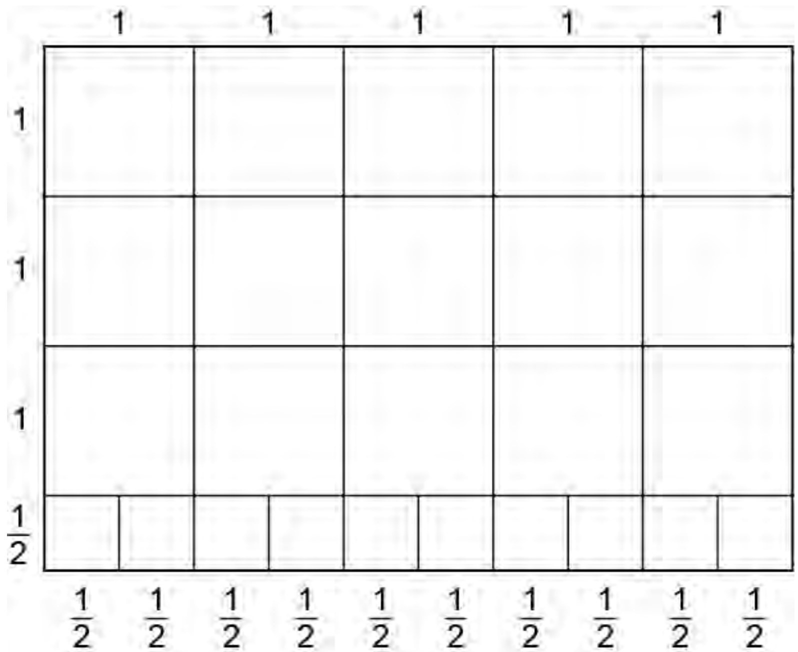
- Ask students to measure, draw, and count tiles to determine the area of a rectangular piece of paper that has at least 1 fractional edge length. For example, provide every student with an index card that is 5 inches by $3\frac{1}{2}$ inches. Then, use a ruler to measure and draw 5 columns that are each 1 inch wide.



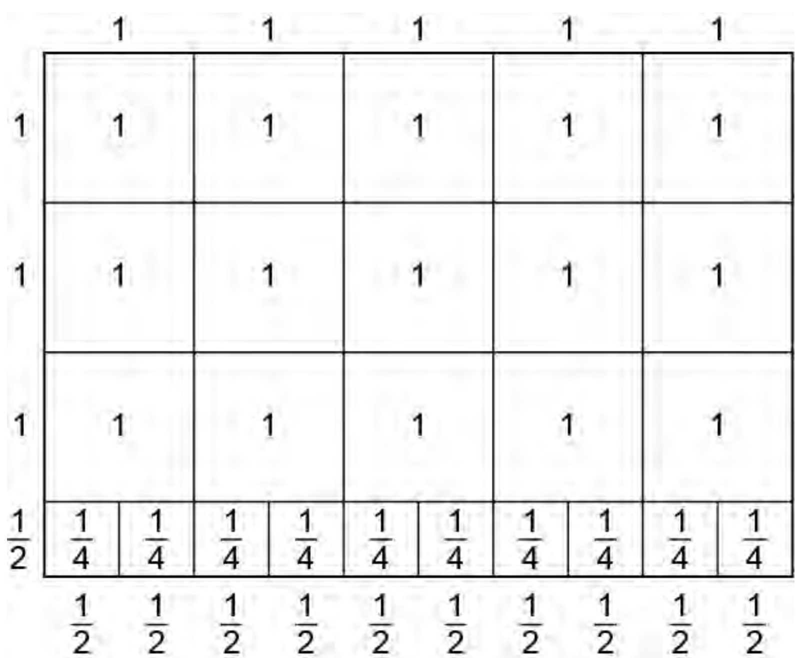
Then, measure and draw 3 rows that are 1 inch in height and 1 row that is $\frac{1}{2}$ inch in height.



Note that in addition to 15 squares with side lengths of 1 inch, there are also 5 rectangles with a length of 1 inch and a width of $\frac{1}{2}$ inch. Divide each rectangle in half, using the ruler to create squares that have side lengths of $\frac{1}{2}$ inch.



Now, write the area of each square.



Finally, combine the areas of all the squares to determine the total area of the rectangle. The total area of the index card is 17.5 square inches because $10 \times \frac{1}{4} = 2\frac{1}{2}$, and $2\frac{1}{2} + 15 = 17\frac{1}{2}$.

Key Academic Terms:

rectangle, area, length, width, fraction, decomposing

Additional Resources:

- Video: Rectangles with fractional side lengths
- Activity: Chavone's bathroom tiles

12d

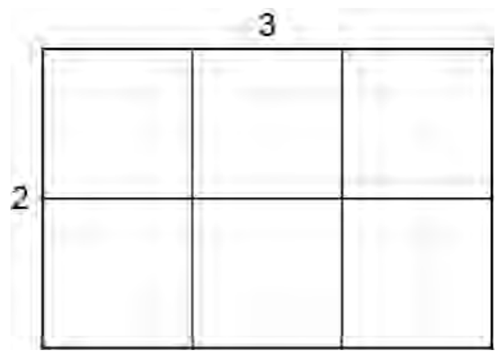
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
12. Apply and extend previous understandings of multiplication to find the product of a fraction times a whole number or a fraction times a fraction. d. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths to show that the area is the same as would be found by multiplying the side lengths.

Guiding Questions with Connections to Mathematical Practices:

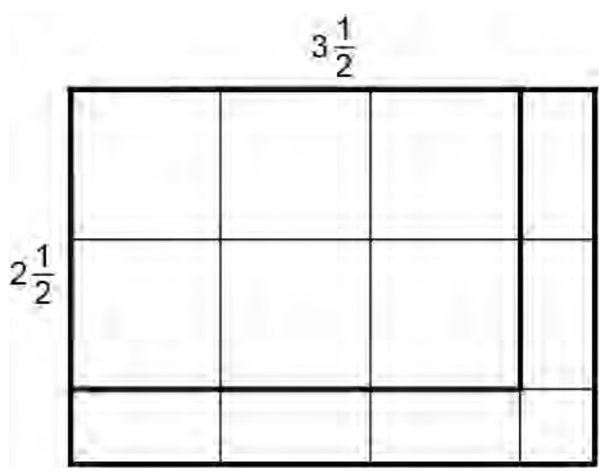
How is the process for determining the area of a rectangle with fractional edge lengths similar to the process for determining the area of a rectangle with whole-number edge lengths?

M.P.4. Model with mathematics. Extend previous knowledge of the formula for the area of a rectangle to rectangles with fractional edge lengths. For example, just as the area of a rectangle with a length of 13 units and a width of 22 units is determined by decomposing 13×22 as $(10 + 3) \times (20 + 2)$, the area for a rectangle with a length of $2\frac{1}{3}$ units and a width of $1\frac{1}{2}$ unit is determined by decomposing $2\frac{1}{3} \times 1\frac{1}{2}$ as $(2 + \frac{1}{3}) \times (1 + \frac{1}{2})$. Additionally, combine the general rule for multiplying fractions (i.e., $(\frac{a}{b}) \times (\frac{c}{d}) = \frac{ac}{bd})$ with the formula for the area of a rectangle to solve real-world problems involving rectangles with fractional edge lengths.

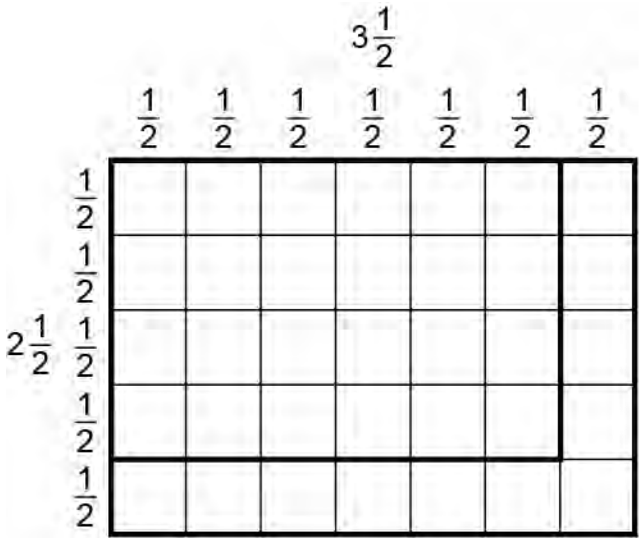
- Ask students to determine the area of a rectangle with whole-number dimensions by tiling it with unit squares. For example, draw a rectangle with a length of 3 units and width of 2 units and then count the tiles to determine the area.



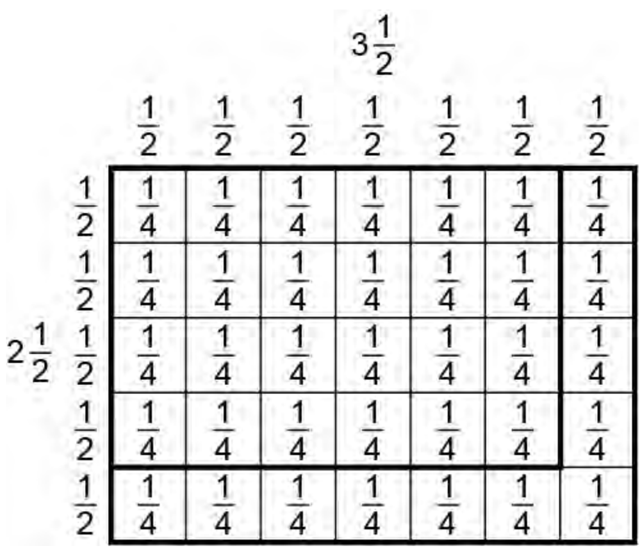
The area is 6 square units. Now, ask students to increase the length of each side by $\frac{1}{2}$ unit and draw the partial tiles.



Next, ask students to further divide the rectangle into squares that all have the same fractional side length. In this case, the rectangle can be tiled with squares that have side lengths of $\frac{1}{2}$ unit.



Note that the area of each square is $\frac{1}{4}$ square units because 4 squares make a unit square.



The squares can now be counted to determine that the area of the new rectangle is $8\frac{3}{4}$ square units because $\frac{1}{4}$ counted 35 times is $8\frac{3}{4}$. Emphasize that while calculating the area of rectangles with whole-number side lengths involves counting squares with whole-number side lengths, calculating the area of rectangles with fractional side lengths involves counting squares with fractional side lengths.

- Ask students to use real-world problems involving rectangles with fractional side lengths using the formula $A = l \times w$ and the general rule $(\frac{a}{b}) \times (\frac{c}{d}) = \frac{ac}{bd}$. For example, “Kathleen sews a patch on her jacket that is $\frac{11}{16}$ inch wide and $\frac{7}{8}$ inch long.” Using the formula $A = l \times w$, write and evaluate the expression $\frac{11}{16} \times \frac{7}{8}$ to determine that the area of the patch is $\frac{77}{128}$ square inches.

Key Academic Terms:

rectangle, area, length, width, fraction, decomposing

Additional Resources:

- Video: Rectangles with fractional side lengths
- Activity: Chavone’s bathroom tiles

13a

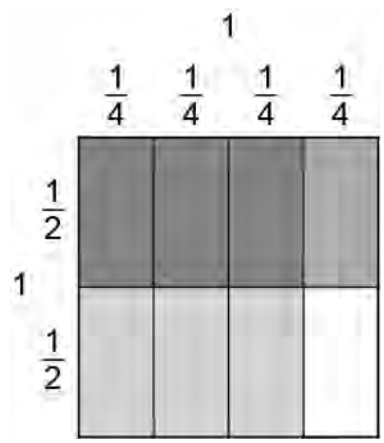
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
13. Interpret multiplication as scaling (resizing). a. Compare the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication. *Example*: *Use reasoning to determine which expression is greater.* 225 *or* $\frac{3}{4} \times 225$; $\frac{11}{50}$ *or* $\frac{3}{2} \times \frac{11}{50}$

Guiding Questions with Connections to Mathematical Practices:

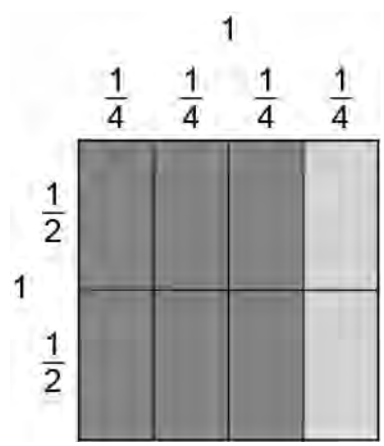
How does the number by which you multiply another number impact the value of the product?

M.P.2. Reason abstractly and quantitatively. Extend previous knowledge of multiplying fractions and equivalence to show that multiplying any fraction by $\frac{n}{n}$ is the same as multiplying by 1. For example, $\frac{3}{4} \times \frac{2}{2} = \frac{6}{8} = \frac{3}{4}$, so multiplying by a fraction with the same numerator and denominator is the same as multiplying by 1. Additionally, use diagrams to visualize why multiplying by the fraction $\frac{n}{n}$ is the same as multiplying by 1.

- Ask students to visually represent the product of two fractions, the second having a numerator that is less than the denominator. Then, ask students to change the second fraction so that the numerator is the same as the denominator and create a new representation. Note that when the second fraction is changed, the product is equal to the first fraction. Lastly, ask students to change the second fraction so that the numerator is greater than the denominator. For example, begin with the problem $\frac{3}{4} \times \frac{1}{2}$ and represent the product visually.

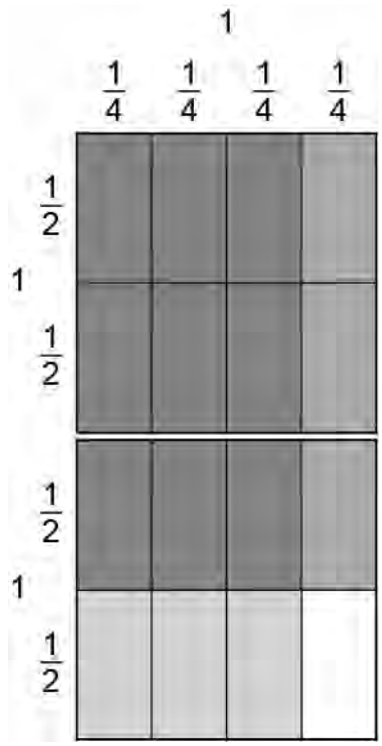


Make note that the product is less than the first fraction, $\frac{3}{4}$. Then, have the students compare this result to a model of $\frac{3}{4} \times \frac{2}{2}$.



In this case, the model shows that the product is $\frac{6}{8}$, or $\frac{3}{4}$, which is the same as the first fraction. Emphasize that when a number is multiplied by a fraction with the same numerator and denominator, it is equivalent to multiplying by 1.

Lastly, ask students to compare these results to $\frac{3}{4} \times \frac{3}{2}$.



In this case, the product is greater than the first fraction.

- Give students several multiplication problems that involve a whole number and a fraction. Then, ask the students to group the problems based on whether the product is larger than its whole number, smaller than its whole number, or the same size as its whole number. For example, begin with the following problems.

$$4 \times \frac{3}{2}$$

$$4 \times \frac{2}{3}$$

$$4 \times \frac{2}{2}$$

$$2 \times \frac{4}{4}$$

$$2 \times \frac{3}{4}$$

$$2 \times \frac{4}{3}$$

After analyzing each problem, determine that $4 \times \frac{3}{2}$ and $2 \times \frac{4}{3}$ result in products larger than the whole number, $4 \times \frac{2}{3}$ and $2 \times \frac{3}{4}$ result in products smaller than the whole number, and $4 \times \frac{2}{2}$ and $2 \times \frac{4}{4}$ result in products the same size as the whole number.

How does a product reflect the sizes of its factors?

M.P.2. Reason abstractly and quantitatively. Know that a product indicates how many times larger or smaller one factor is compared to the other as a multiplicative comparison. For example, the product of $\frac{1}{3}$ and 12 is 4, which indicates that 4 is $\frac{1}{3}$ the size of 12. Additionally, explain the meaning of a multiplication equation in various ways using comparative language.

- Ask students to use a multiplication equation to write sentences that compare the product size to the size of the factors. For example, begin with the equation $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. Then, write comparative statements such as "$\frac{1}{6}$ is $\frac{1}{2}$ of $\frac{1}{3}$" and "$\frac{1}{6}$ is $\frac{1}{3}$ of $\frac{1}{2}$."
- Ask students to use a comparative statement to write a corresponding equation. For example, begin with the statement "9 is $\frac{1}{5}$ of 45." Then, represent the comparative statement with the equations $9 = \frac{1}{5} \times 45$ or $\frac{1}{5} \times 45 = 9$.

Key Academic Terms:

factor, product, resizing, scaling, equivalent, equation

Additional Resources:

- Lesson: [Making sense of multiplying fractions]
- Activity: [Comparing a number and a product]

13b

Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
13. Interpret multiplication as scaling (resizing). b. Explain why multiplying a given number by a fraction greater than 1 results in a product greater than the given number and relate the principle of fraction equivalence.

Guiding Questions with Connections to Mathematical Practices:

How does multiplying by a fraction greater than 1 impact the value of the product?

M.P.2. Reason abstractly and quantitatively. Extend previous knowledge of multiplying two whole numbers to multiplying a whole number and a fraction greater than 1. For example, just as the equation $2 \times 3 = 6$ indicates that 6 is 3 times as great as 2, the equation $2 \times \frac{3}{2} = 3$ indicates that 3 is $\frac{3}{2}$ (i.e., 1.5) times as great as 2. Additionally, analyze a multiplication equation or inequality to determine the value of an unknown factor.

- Ask students to identify a missing symbol in a mathematical statement and explain the reasoning. For example, begin with the statement shown.

  $$10 \times \frac{8}{7} \square 10$$

  Identify that the missing symbol is $>$, making the statement $10 \times \frac{8}{7} > 10$. Explain that the product must be greater than 10 because 10 is multiplied by a fraction that is greater than 1.

- Ask students to analyze an inequality that involves the product of a whole number and a fraction and identify information about one of the factors. For example, provide students with the inequality $4 \times \frac{?}{7} > 4$. Then, find a single-digit whole number that makes the inequality true. In this case, the missing numerator could be either 8 or 9 because those are the only two single digits that create a fraction greater than 1. Note that if numbers other than single-digit numbers were allowed, there would be other possible correct answers.

Key Academic Terms:

factor, product, equivalent, resizing, equation, scaling

Additional Resources:

- Lesson: Making sense of multiplying fractions
- Activity: Mrs. Gray's homework assignment

13c

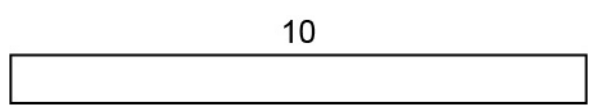
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
13. Interpret multiplication as scaling (resizing). c. Explain why multiplying a given number by a fraction less than 1 results in a product smaller than the given number and relate the principle of fraction equivalence.

Guiding Questions with Connections to Mathematical Practices:

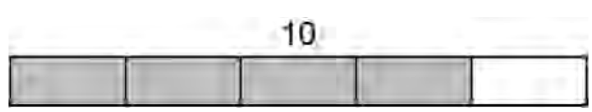
How does multiplying by a fraction less than 1 impact the value of the product?

M.P.2. Reason abstractly and quantitatively. Extend previous knowledge of multiplying two whole numbers to multiplying a whole number and a fraction less than 1. For example, just as the equation $8 \times 4 = 32$ indicates that 32 is 4 times as great as 8, the equation $8 \times \frac{3}{4} = 6$ indicates that 6 is $\frac{3}{4}$ the size of 8. Additionally, create diagrams to illustrate why multiplying a whole number by a fraction less than 1 results in a product that is less than the whole number.

- Ask students to make an illustration that represents the size of the product relative to the whole number when given a whole number and a fraction less than 1. For example, begin with the numbers 10 and $\frac{4}{5}$ and represent the number 10 with a tape diagram.



Then, divide the diagram into 5 equal sections with 4 of the sections shaded.



Note how the shaded sections representing $\frac{4}{5}$ of 10 compare to the whole number 10.

- Ask students to make inferences from a general mathematical statement involving the product of a whole number and a fraction. For example, begin with the mathematical sentence $p \times \frac{q}{r} < p$, where each letter represents a single-digit whole number. Make observations on how the size of q and the size of r relate. In this case, q must be less than r because $\frac{q}{r}$ must be a fraction less than 1 for the statement to be true.

Key Academic Terms:

factor, product, equivalent, resizing, equation, scaling

Additional Resources:

- Lesson: Making sense of multiplying fractions
- Activity: Mrs. Gray's homework assignment

14

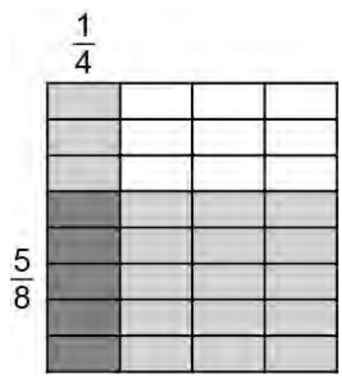
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
14. Model and solve real-world problems involving multiplication of fractions and mixed numbers using visual fraction models, drawings, or equations to represent the problem.

Guiding Questions with Connections to Mathematical Practices:

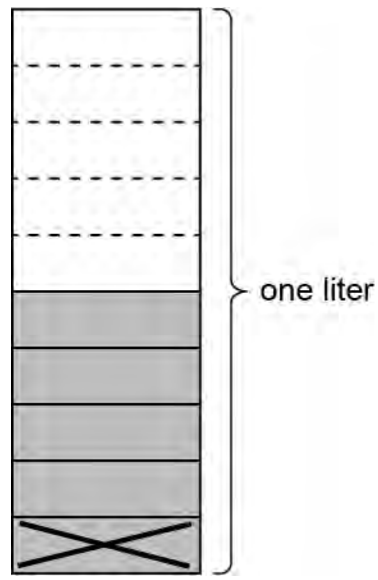
How can the product of two fractions in context be visualized?

M.P.4. Model with mathematics. Use fraction models to visually represent the product of two fractions in context. For example, if $\frac{1}{2}$ of the snacks in a box are granola bars, and $\frac{1}{4}$ of the granola bars have chocolate chips, then a 2×4 rectangular grid can be constructed with $\frac{1}{2}$ of the pieces (i.e., 4) shaded to represent granola bar snacks and $\frac{1}{4}$ of those shaded pieces (i.e., 1) marked with a c to represent which ones have chocolate chips. This demonstrates that $\frac{1}{8}$ of the snacks in the box are chocolate chip granola bars. Additionally, bar models may be used to show the products of fractions and/or mixed numbers.

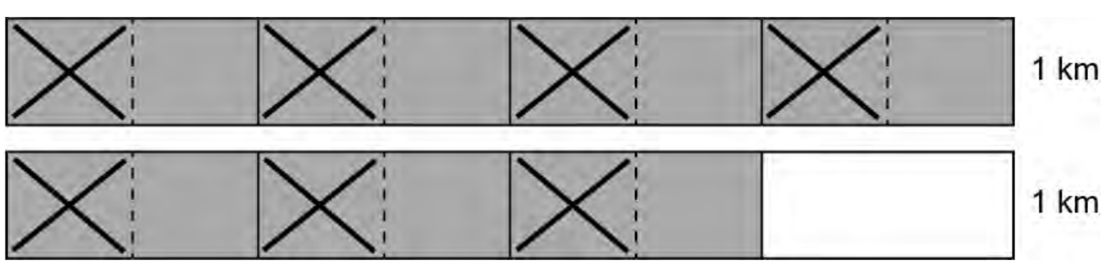
- Ask students to construct a grid model to determine the product of two fractions. For example, “Avery has $\frac{5}{8}$ of a package of paper and he wants to give his friends each $\frac{1}{4}$ of the paper that he has. What fraction of the package of paper does each friend receive?” Beginning with a rectangle, divide the grid vertically into 4 columns and horizontally into 8 rows. Then, shade one of the fourths columns and five of the eighths rows. The overlapped sections indicate the product $\frac{1}{4} \times \frac{5}{8} = \frac{5}{32}$.



- Ask students to construct a bar model to determine the product of 2 fractions. For example, give students the scenario "Marcus has $\frac{1}{2}$ of a 1-liter bottle of water and pours it evenly into 5 glasses." Construct a bar model that is shaded for the $\frac{1}{2}$ liter, then divide the $\frac{1}{2}$ into fifths and place an X in one of the fifths to determine that each glass holds $\frac{1}{10}$ of a liter of water: $\frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$.



- Ask students to construct a model to determine the product of a fraction and a mixed number. For example, if Eve lives $1\frac{3}{4}$ kilometers from school and she runs $\frac{1}{2}$ of the way to school, create two bars split into fourths and use shading to show $1\frac{3}{4}$. Then, split each shaded fourth in half and place an X in every other half to indicate that Eve ran $\frac{7}{8}$ of a kilometer.



Why is it important to attend to the meaning of the underlying quantities in a fraction multiplication word problem?

M.P.2. Reason abstractly and quantitatively. Analyze the problem to determine the meaning of the solution. For example, given the solution of $\frac{1}{8}$ for the granola bar problem in the guiding question above, observe that because the initial number of snacks is not given, only the fraction of the snacks that are chocolate chip granola bars is known, not the total number of chocolate chip granola bars. Additionally, if the solution to a problem is $\frac{1}{8}$ of a specified quantity like gallons, the fraction has units of gallons.

- Ask students to interpret the meaning of a product of fractions in context where no specified units are given. For example, "A baker purchases $\frac{1}{2}$ barrel of flour, then takes that $\frac{1}{2}$ and splits it evenly to bake 6 cakes so that each cake uses $\frac{1}{6}$ of the baker's flour." The multiplication of $\frac{1}{2} \times \frac{1}{6}$ indicates the fraction of the barrel of flour that is in each cake. Additionally, if a student has $\frac{1}{5}$ of a package of new pencils and gives $\frac{1}{3}$ of those pencils to a friend, the interpretation of $\frac{1}{5} \times \frac{1}{3}$ would be $\frac{1}{15}$ of the package of pencils, not a specified quantity of pencils.

- Ask students to interpret the meaning of a product of fractions in context where specified units are given. For example, "A relay team of 4 students is running a $\frac{1}{2}$-mile race such that each student is running $\frac{1}{4}$ of $\frac{1}{2}$ of a mile." The product of these two fractions indicates that each person on the team is running $\frac{1}{8}$ of a mile.

Key Academic Terms:

fraction, mixed number, multiplication

Additional Resources:

- Activity: [Fraction word problems: Do I multiply or divide?]
- Activity: [Comparing heights of buildings]

15a

Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
15. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. a. Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions and illustrate using visual fraction models, drawings, and equations to represent the problem.

Guiding Questions with Connections to Mathematical Practices:

How can multiplication and the meaning of division be used to solve division problems with fractions?

M.P.2. Reason abstractly and quantitatively. Know that a division problem can be expressed as a multiplication problem with a missing factor. For example, the equation $10 \div \frac{1}{2} = \square$ can be rewritten as $\square \times \frac{1}{2} = 10$ and expressed as "How many $\frac{1}{2}$s are in 10?" Additionally, connect this knowledge to whole number reasoning, such as $12 \div \frac{1}{4} = \square$ can be rewritten as $\square \times \frac{1}{4} = 12$, just as $10 \div 5 = \square$ can be rewritten as $\square \times 5 = 10$.

- Ask students to use the meaning of multiplication and division and whole number reasoning to write equivalent equations. For example, list the multiplicative relationships between 6, 8, and 48 as $6 \times 8 = 48$, $8 \times 6 = 48$, $48 \div 8 = 6$, and $48 \div 6 = 8$. Students should then generalize the multiplicative relationships to multiplying unit fractions and whole numbers, such as $8 \div \frac{1}{6} = \square$, $8 \div \square = \frac{1}{6}$, $\frac{1}{6} \times \square = 8$, $\square \times \frac{1}{6} = 8$. Use the form that makes the most sense to solve each problem and use strategies like the meaning of multiplication or visual models.

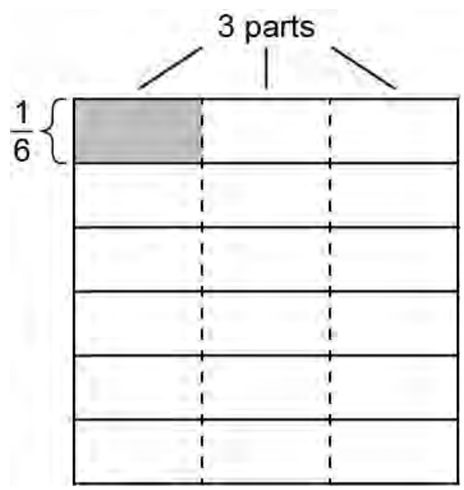
- Ask students to use multiplicative language to describe division situations with fractions. For example, describe $\frac{1}{5} \div 4$ as "one-fifth of a whole divided into four equal parts" or $9 \div \frac{1}{2}$ as "the number of one-half pieces that make up nine wholes." Students should determine the unit, or whole, whenever dividing with fractions.

M.P.8. Look for and express regularity in repeated reasoning. Connect previous knowledge of fractions as division to the idea of multiplying a number by a unit fraction to solve division problems. For example, since $5 \div 4 = 5 \times \frac{1}{4}$, then $\frac{1}{5} \div 4 = \frac{1}{5} \times \frac{1}{4} = \frac{1 \times 1}{5 \times 4} = \frac{1}{20}$. Additionally, since $10 \div 3 = 10 \times \frac{1}{3}$, then $\frac{1}{10} \div 3 = \frac{1}{10} \times \frac{1}{3} = \frac{1 \times 1}{10 \times 3} = \frac{1}{30}$.

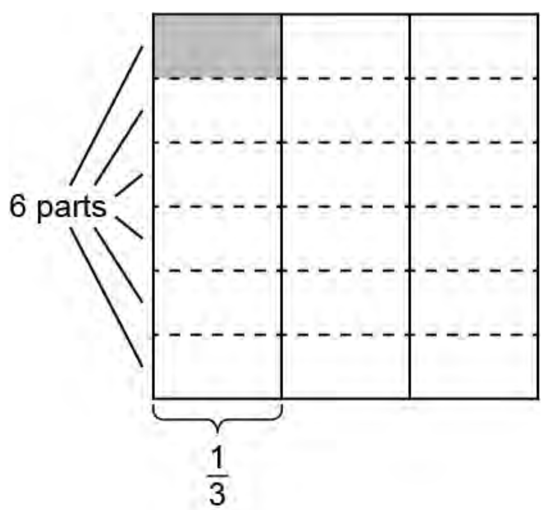
- Ask students to connect multiplication and division with whole numbers to multiplication and division with fractions by extending fact families. For example, the facts of $3 \times 5 = 15$, $5 \times 3 = 15$, $15 \div 3 = 5$, and $15 \div 5 = 3$ can be extended to include $15 \times \frac{1}{3} = 5$ because the fraction $\frac{15}{3}$ can be interpreted both as a quotient of the numerator and denominator, $15 \div 3$, and equivalently as a multiple of a unit fraction, $15 \times \frac{1}{3}$. Therefore, the fact $15 \div 3 = 5$ is equivalent to $15 \times \frac{1}{3} = 5$. Similarly, the fact family can also be extended to include $15 \times \frac{1}{5} = 3$. Further, these two new multiplication equations imply the additional division equations $3 \div \frac{1}{5} = 15$ and $5 \div \frac{1}{3} = 15$.

- Ask students to extend their knowledge of fractions to explain why dividing by a nonzero whole number is equivalent to multiplying by its unit fraction. For example, the quotient $8 \div 3$ is equivalent to the fraction $\frac{8}{3}$, which in turn is equivalent to $8 \times \frac{1}{3}$. Similarly, the quotient $\frac{1}{2} \div 3$ would be equivalent to the fraction $\frac{\frac{1}{2}}{3}$, which in turn would be equivalent to $\frac{1}{2} \times \frac{1}{3}$.

- Ask students to use visual models to explain why dividing by a nonzero whole number is equivalent to multiplying by its unit fraction. For example, ask students to compare $\frac{1}{6} \div 3$ and $\frac{1}{6} \times \frac{1}{3}$. The first expression starts with $\frac{1}{6}$ of a whole and divides it into 3 equal portions, as shown.



The expression $\frac{1}{6} \times 8$ means "one part of a partition of 8 into 6 equal parts." Similarly, $\frac{1}{6} \times \frac{1}{3}$ means "one part of a partition of $\frac{1}{3}$ into 6 equal parts." This can be modeled visually, as shown.

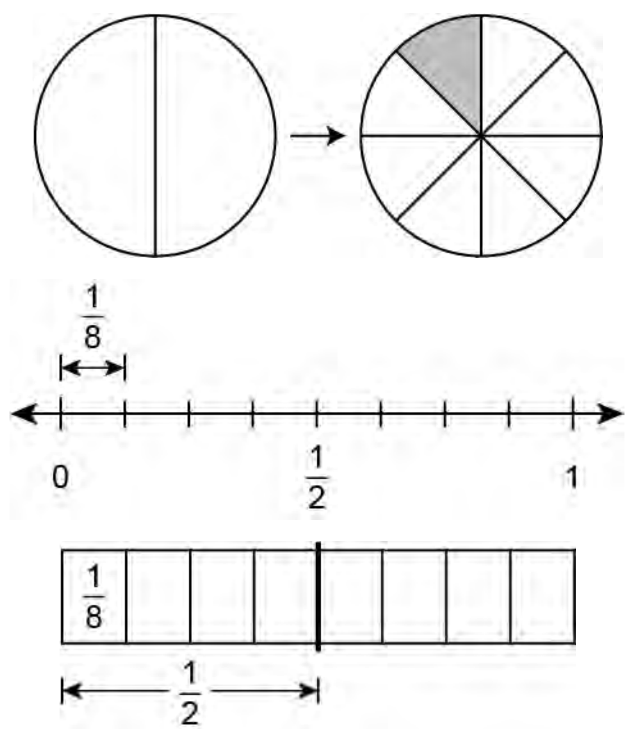


Note that both models give identical results.

How can the quotient of a unit fraction and a whole number be represented visually?

M.P.4. Model with mathematics. Create fraction models to represent quotients of unit fractions and whole numbers, using context to make meaning of the whole. For example, to represent $\frac{1}{2} \div 4$, write the story problem “Mike has $\frac{1}{2}$ of a foot of ribbon. He cuts the ribbon into 4 equal pieces. How many feet of ribbon is each portion of the ribbon after he cuts it?” Then, represent $\frac{1}{2}$ on a number line that begins at 0 and ends at 1. To represent division by 4, both halves of the number line are split into 4 equal sections to show the fraction of the $\frac{1}{2}$ of a foot of ribbon that Mike cut. The quotient $\frac{1}{2} \div 4$ is represented by 1 of the sections of the whole foot, which is located at $\frac{1}{8}$ on the number line. Additionally, use an area model to represent the context by drawing a rectangle, using a vertical line to show the halves and 3 horizontal lines to show the fourths. Observe that $\frac{1}{8}$ of the rectangle represents one portion of ribbon.

- Ask students to draw a visual model or use manipulatives to represent a division problem involving fractions. For example, represent $\frac{1}{2} \div 4$ visually. Three possible ways are shown.



Key Academic Terms:

unit fraction, whole number, division, multiplication, factor, number line

Additional Resources:

- Game: Dividing unit fractions by whole numbers
- Activity: Painting a room

15b

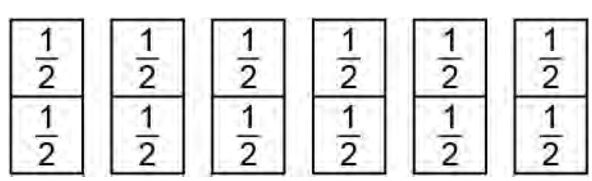
Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
15. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. b. Create a story context for a unit fraction divided by a whole number, and use a visual fraction model to show the quotient.

Guiding Questions with Connections to Mathematical Practices:

How can real-world problems involving the quotients of unit fractions and whole numbers be solved?

M.P.4. Model with mathematics. Utilize fraction models and equations to solve real-world problems that involve division with unit fractions. For example, if 1 pizza contains 3 servings, then a fraction model can be used to determine the total number of servings in 4 pizzas. More specifically, 4 circles that are each divided into thirds can be used to show that $4 \div \frac{1}{3} = 12$ because there are 12 sections, each containing $\frac{1}{3}$ of a pizza, in the 4 circles. Additionally, on a number line, cut each section of 1 (to represent 1 pizza) into 3 equal pieces for a total of 12 pieces.

- Ask students to use models to represent and solve real-world problems. For example, given the problem "It takes about $\frac{1}{2}$ gallon of paint to paint a wall in Samir's home. About how many walls did Samir paint if he used 6 gallons of paint?" make a representation like the one shown.



- Ask students to ensure an answer is reasonable by using generalizations formed about fraction and whole-number division and multiplication. For example, students should note that a unit fraction divided by a whole number always results in a quotient that is less than the unit fraction, unless the whole number is 1.
- Ask students to write a context to make meaning from a fraction division problem. For example, given the expression $\frac{1}{4} \div 10$, write the context "Melanie is making greeting cards. She uses $\frac{1}{4}$ of a sheet of construction paper to make into decorations for her 10 greeting cards. What fraction of a sheet of construction paper is each greeting card going to have if she uses the same amount for each card?"

Key Academic Terms:

unit fraction, whole number, division, fraction model

Additional Resources:

- Activity: How many marbles?
- Activity: Salad dressing

15c

Operations with Numbers: Fractions
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
15. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. c. Create a story context for a whole number divided by a unit fraction, and use a visual fraction model to show the quotient.

Guiding Questions with Connections to Mathematical Practices:

Why is the quotient of a whole number and a unit fraction larger than the dividend?

M.P.2. Reason abstractly and quantitatively. Use the meaning of division and the relationship between multiplication and division to demonstrate that the quotient of a whole number and a unit fraction is larger than the dividend because the divisor is separating the dividend into fractional pieces, and the quotient represents the number of pieces into which the dividend has been divided. For example, the expression $4 \div \frac{1}{3}$ is asking how many $\frac{1}{3}$ pieces are in 4 wholes. Additionally, using the relationship between multiplication and division, demonstrate that dividing a whole number by a unit fraction is the same as multiplying two whole numbers, e.g., $4 \div \frac{1}{3} = 4 \times 3$.

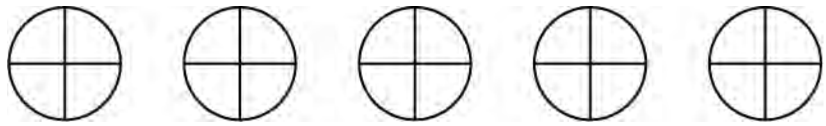
- Ask students to extend patterns of division. For example, ask students to compare the quotients $48 \div 12 = 4$, $48 \div 8 = 6$, $48 \div 3 = 16$, and $48 \div 1 = 48$. Help students to observe that as the divisor decreases, the quotient increases. Further, help students extend this pattern to conclude that dividing by a proper fraction would result in an even larger quotient.

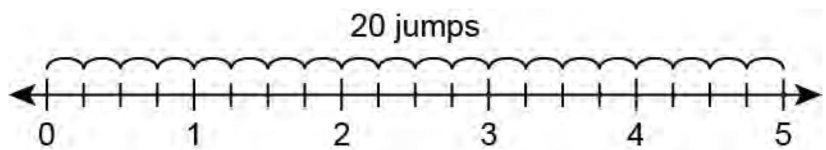
- Ask students to use a partitioning model of division to explain why dividing by a unit fraction gives a result that is greater than the dividend. For example, give students the quotient $8 \div \frac{1}{2}$. Division of 8 by a number can be modeled by the scenario "Jordyn has 8 cookies and wants to share the same number of cookies with each of her friends. How many friends can share the cookies?" If each friend receives 1 cookie, then the cookies can be shared among 8 friends. If each friend receives $\frac{1}{2}$ cookie, then the cookies can be shared among more than 8 friends because each portion is smaller.

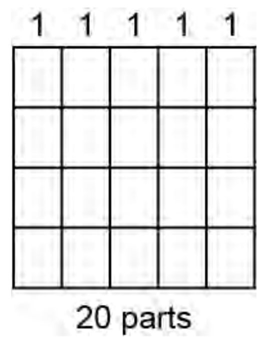
How can the quotient of a whole number and a unit fraction be represented visually?

M.P.4. Model with mathematics. Create fraction models to represent quotients of whole numbers and unit fractions. For example, the quotient of 2 and $\frac{1}{3}$ can be represented with 2 rectangles that are each divided into thirds, yielding a total of $\frac{6}{3}$ within the two original rectangles. As such, $2 \div \frac{1}{3} = 6$. Additionally, $2 \div \frac{1}{3}$ can be represented on a number line by splitting the number line for each whole, which is 1, into 3 pieces for a total of 6 pieces.

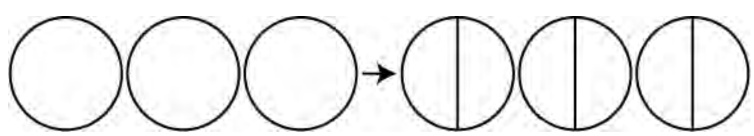
- Ask students to use a visual model to represent equations and expressions written as a whole number divided by a unit fraction. For example, the expression $5 \div \frac{1}{4}$ may be represented in a variety of ways, as shown.







- Ask students to write an equation from a visual model. For example, given the visual model shown, write a true equation that accurately represents the model, such as $3 \div \frac{1}{2} = 6$.



How can context be used to interpret division problems involving fractions?

M.P.2. Reason abstractly and quantitatively. Write a story problem to make sense of a division problem involving fractions. For example, given the equation $6 \div \frac{1}{4} = 24$, write the story problem "6 cups of cat food will provide 24 servings of $\frac{1}{4}$ of a cup per serving." Additionally, connect the context to a visual representation to solve the problem.

- Ask students to write a story problem for a division problem involving fractions. For example, given the expression $8 \div \frac{1}{6}$, write the story problem "Julius has 8 jars of pickles. A serving of pickles is $\frac{1}{6}$ of the jar. How many servings of pickles does Julius have in all of the jars?"

- Ask students to connect a fraction division problem, context, and a visual model. For example, given the division problem $7 \div \frac{1}{2}$, write the story problem "There are 7 cups of oatmeal. A single serving of oatmeal is $\frac{1}{2}$ cup. How many servings of oatmeal are there?" and draw a visual model like the one shown.

Students should also connect all three of these (i.e., the fraction division problem, context, and visual model) to the equivalence between $7 \div \frac{1}{2}$ and 7×2.

Key Academic Terms:

unit fraction, whole number, division, dividend, divisor, quotient, equation, rectangle, fractional pieces

Additional Resources:

- Game: [Dividing whole numbers by unit fractions]
- Video: [Dividing whole #'s by fractions]

16a

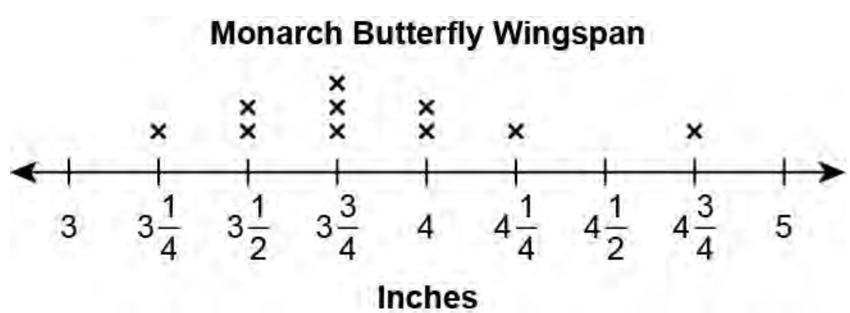
Data Analysis
Represent and interpret data.
16. Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). a. Add, subtract, multiply, and divide fractions to solve problems involving information presented in line plots. *Note: Division is limited to unit fractions by whole numbers and whole numbers by unit fractions.*

Guiding Questions with Connections to Mathematical Practices:

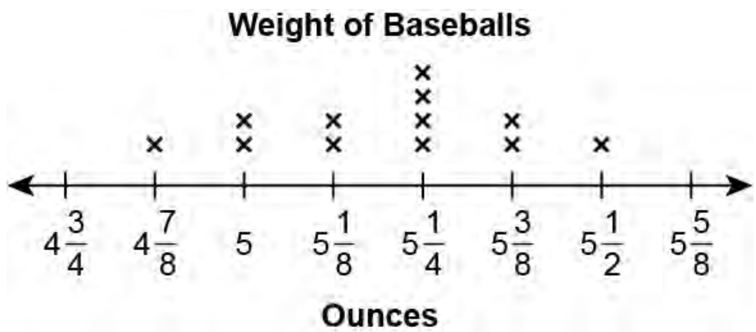
How is a line plot constructed when data measurements include fractions of a unit?

M.P.5. Use appropriate tools strategically. Represent the frequency of data by creating a number line with fractional intervals and placing an X above the locations that indicate each data value. For example, if a data set includes data points $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1, then a number line with tick marks at $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, and $\frac{4}{4}$ can be created with 1 X above $\frac{1}{4}$, 2 Xs stacked vertically above $\frac{2}{4}$, 1 X above $\frac{3}{4}$, and 1 X above $\frac{4}{4}$. Additionally, line plots always contain uniform intervals, just like number lines.

- Ask students to make a line plot to represent a given data set in which the data are listed in numerical order. For example, make a line plot to represent the following data set, which shows the lengths of 10 Monarch butterfly wingspans, each rounded to the nearest quarter of an inch: $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{1}{2}$, $3\frac{3}{4}$, $3\frac{3}{4}$, $3\frac{3}{4}$, 4, 4, $4\frac{1}{4}$, $4\frac{3}{4}$.



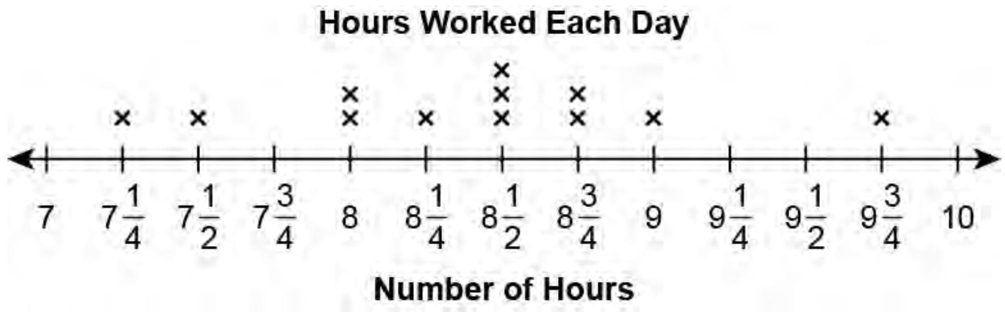
- Ask students to make a line plot to represent a given data set in which the data are not listed in numerical order. For example, make a line plot to represent the following data set, which shows the weights of 12 baseballs, each rounded to the nearest $\frac{1}{8}$ ounce: $4\frac{7}{8}$, $5\frac{1}{8}$, $5\frac{1}{4}$, $5\frac{1}{4}$, 5, $5\frac{3}{8}$, $5\frac{1}{8}$, $5\frac{1}{4}$, $5\frac{1}{2}$, $5\frac{1}{4}$, 5, $5\frac{3}{8}$.



How can information about data presented in line plots be determined using the four operations?

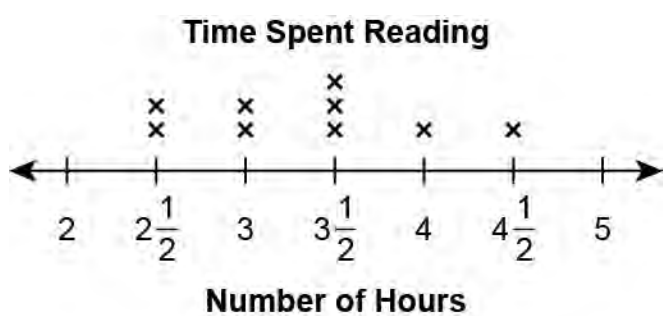
M.P.8. Look for and express regularity in repeated reasoning. Apply knowledge of addition, subtraction, multiplication, and division to determine information about a set of data displayed on a line plot. For example, if the lengths, in inches, of some insects are shown on a line plot to be $\frac{3}{4}$, $\frac{5}{2}$, $\frac{7}{8}$, $\frac{8}{8}$, and $\frac{9}{4}$, then it can be determined that the total length of all 5 insects is $7\frac{3}{8}$ inches because $\frac{3}{4}+\frac{5}{2}+\frac{7}{8}+\frac{8}{8}+\frac{9}{4}=7\frac{3}{8}$. Additionally, if the total value of all the measurements is known, that information can be used to find missing data points.

- Ask students to find the difference between the greatest and least values on a given line plot. For example, the line plot shown represents the number of hours an employee has worked in the past 12 days.



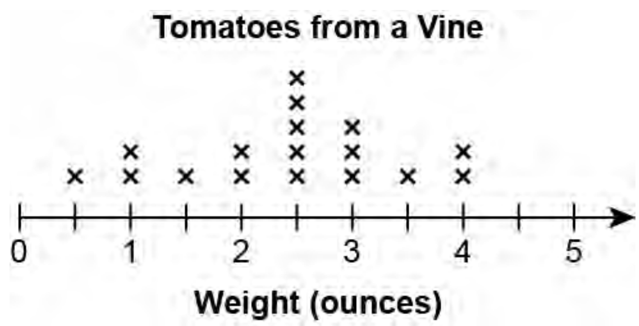
The greatest value in the line plot is $9\frac{3}{4}$, and the least value is $7\frac{1}{4}$, so the difference is $2\frac{1}{2}$ hours because $9\frac{3}{4} - 7\frac{1}{4} = 2\frac{2}{4}$.

- Ask students to identify the missing data point in a given line plot when the total value of all the measurements is known. For example, give students the prompt "A class of 10 students recorded the number of hours each student spent reading this week. The results for 9 of the students are shown in the line plot. The 10 students combined read a total of $32\frac{1}{2}$ hours this week. Find the missing value that represents the number of hours the tenth student read to make a total of $32\frac{1}{2}$ hours."



The total of the 9 values shown is 30 hours because adding together the values gives $2\frac{1}{2} + 2\frac{1}{2} + 3 + 3 + 3\frac{1}{2} + 3\frac{1}{2} + 3\frac{1}{2} + 4 + 4\frac{1}{2} = 27\frac{6}{2} = 30$. To find the missing value, subtract 30 hours from the given total of $32\frac{1}{2}$ hours. Therefore, the missing X should be above $2\frac{1}{2}$ because $32\frac{1}{2} - 30 = 2\frac{1}{2}$.

- Ask students to create a multiplicative comparison of the largest and the smallest values shown in a line plot. For example, the following line plot shows the weights of the tomatoes picked from a vine.



The heaviest tomato weighs 4 ounces, and the lightest tomato weighs $\frac{1}{2}$ ounce. Because $4 \div \frac{1}{2} = 8$, the heaviest tomato weighs 8 times as much as the lightest tomato.

Key Academic Terms:

line plot, frequency, fraction, operation, data, number line, fractional intervals

Additional Resources:

- Video: Fractions on a line plot
- Lesson: Learning about line plots

17

Measurement
Convert like measurement units within a given measurement system.
17. Convert among different-sized standard measurement units within a given measurement system and use these conversions in solving multi-step, real-world problems.

Guiding Questions with Connections to Mathematical Practices:

What is the general process for converting from a larger unit of measurement to a smaller unit of measurement within the same system?

M.P.8. Look for and express regularity in repeated reasoning. Know that multiplication can be used to convert a larger unit of measurement to a smaller unit of measurement by multiplying the given measurement by the ratio of the smaller unit of measurement to the larger unit. The size of the given measurement has not changed; only the units have changed. The size of the unit has decreased, but the number of units has increased. For example, when converting a given number of feet into inches, the number of feet is multiplied by 12 because there are 12 inches in every foot. So, to convert 3.5 feet into inches, 3.5 is multiplied by 12. Therefore, there are 42 inches in 3.5 feet because $3.5 \times 12 = 42$. Additionally, when converting to a smaller unit of measurement within the metric system, a given measurement is always multiplied by a power of 10.

- Ask students to convert a measurement from meters to centimeters. For example, convert 7.6 meters to centimeters. Since there are 100 centimeters in a meter, multiply 7.6 by 100 to find that there are 760 centimeters in 7.6 meters.

- Ask students to convert a measurement from kilometers to meters. For example, convert 12 kilometers to meters. Since there are 1,000 meters in a kilometer, multiply 12 by 1,000 to find that there are 12,000 meters in 12 kilometers.

- Ask students to convert a measurement from pounds to ounces. For example, convert 5 pounds to ounces. Since there are 16 ounces in a pound, multiply 5 by 16 to find that there are 80 ounces in 5 pounds.

What is the general process for converting from a smaller unit of measurement to a larger unit of measurement within the same system?

M.P.8. Look for and express regularity in repeated reasoning. Know that division can be used to convert a smaller unit of measurement to a larger unit of measurement by dividing the given measurement by the ratio of the smaller unit of measurement to the larger unit. The size of the given measurement has not changed; only the units have changed. The size of the unit has increased, but the number of units has decreased. For example, when converting 72 ounces into cups, 72 should be divided by 8 because there are 8 ounces in every cup. Therefore, there are 9 cups in 72 ounces because $72 \div 8 = 9$. Additionally, when converting to a larger unit of measurement within the metric system, a given measurement is always divided by a power of 10.

- Ask students to convert a measurement from millimeters to centimeters. For example, convert 6.3 millimeters to centimeters. Since there are 10 millimeters in a centimeter, divide 6.3 by 10 to find that the measurement of 6.3 millimeters is equal to 0.63 centimeters.

- Ask students to convert a measurement from grams to kilograms. For example, convert 8 grams to kilograms. Since there are 1,000 grams in a kilogram, divide 8 by 1,000 to find that the measurement of 8 grams is equal to 0.008 kilogram.

- Ask students to convert a measurement from feet to yards. For example, convert 18 feet to yards. Since there are 3 feet in a yard, divide 18 by 3 to find that the measurement of 18 feet is equal to 6 yards.

How can models be used to represent and solve multistep, real-world problems?

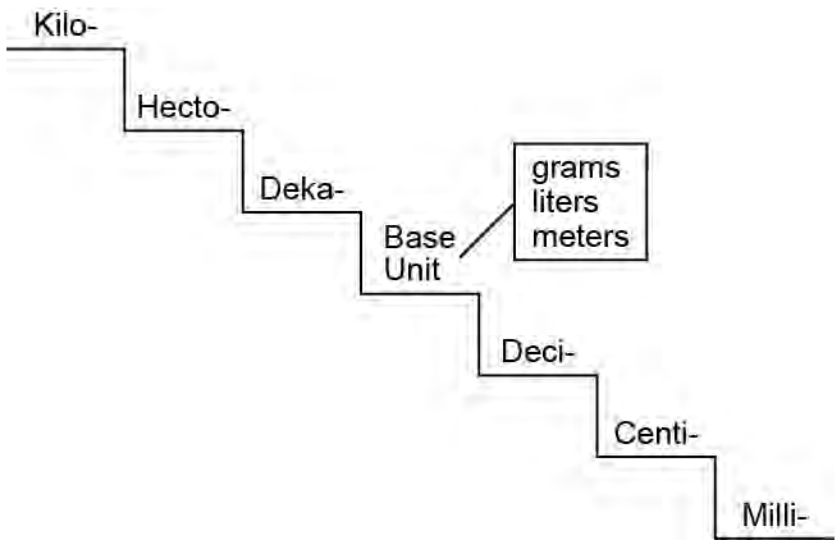
M.P.4. Model with mathematics. Solve multistep, real-world problems using a variety of strategies, including models. For example, create a table of equivalent measurements to solve a multistep problem, such as "Tanya is in a race that is 5 kilometers long. So far, she has run 1,500 meters. How many kilometers does she have left in the race?" Additionally, use visual models such as a metric stair step or graphic organizer to help solve a multistep problem involving unit conversions.

- Ask students to use a conversion table to help solve a multistep problem. For example, the conversion table shown is for the US Customary system for length. Use the table to help solve the problem "A neighborhood has a road that is 2 miles long. There are fire hydrants along the road that are spaced 800 feet apart. How many fire hydrants are along the road?" First, use the conversion table to find how many feet are in 2 miles.

Unit	**Equivalent Unit**
1 mile	5,280 feet
1 yard	3 feet
1 foot	12 inches

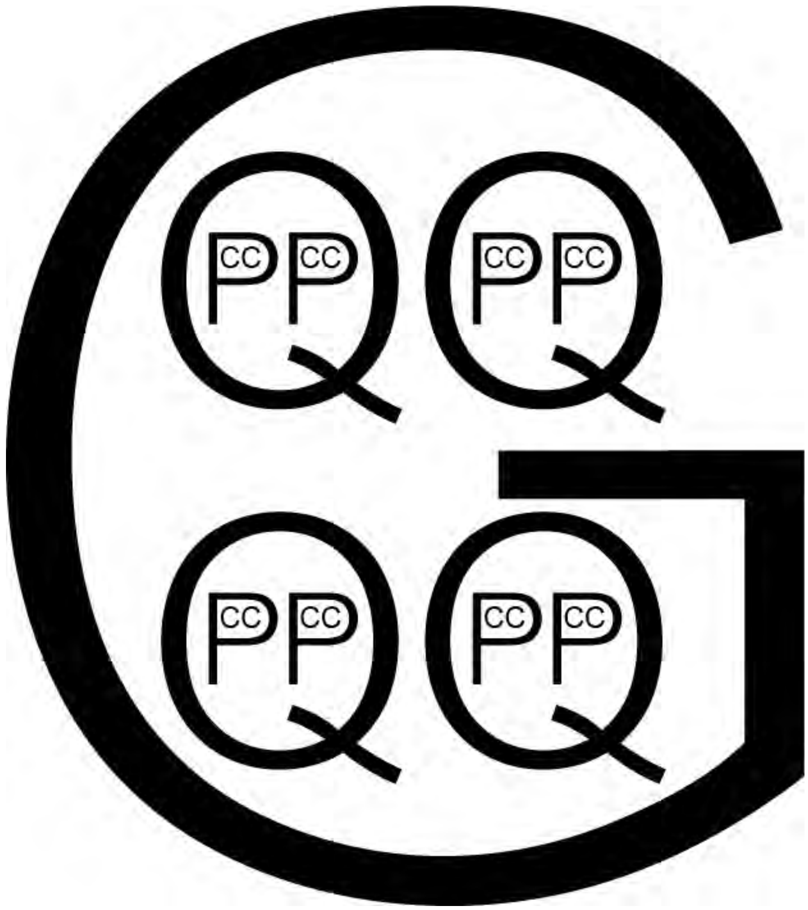
Since there are 5,280 feet in 1 mile, there are $2 \times 5{,}280 = 10{,}560$ feet in 2 miles. Then, to find the number of fire hydrants, divide 10,560 feet by 800 feet to find the quotient of 13.2. Since each fire hydrant is a full 800 feet apart, the remainder will not yield an additional fire hydrant. Therefore, the road has 13 fire hydrants, not including the fire hydrant at the start of the road (distance of 0 feet).

- Provide students with a metric stair step and ask them to solve a multistep problem involving the metric system. For example, give students the prompt “Mariel is growing watermelons. Early in the growing season, the mass of a watermelon was 2.6 kilograms. Four weeks later, the mass of the same watermelon was 4.4 kilograms. If the same rate of growth continues, how many more grams can Mariel expect the mass of the watermelon to gain in the next 2 weeks?” First, find the change in the first 4 weeks by subtracting 2.6 kilograms from 4.4 kilograms. In the first 4 weeks, the watermelon gained 1.8 kilograms. So, in the next 2 weeks, the watermelon will gain half of that amount, or 0.9 kilogram. Use the metric stair step to convert 0.9 kilogram into grams to solve the problem.



On the metric stair step, each step down requires multiplication of 10. Since grams is 3 steps down from kilograms, students need to multiply by 10 three times, which is the same as multiplying by 1,000. Therefore, multiply 0.9 kilogram by 1,000 to conclude that the watermelon is expected to gain 900 grams in the next 2 weeks.

- Ask students to use a graphic organizer to model and help solve a multistep problem. For example, to convert between units of the US Customary system for capacity (gallon, quart, pint, and cup), create a graphic organizer like the figure shown.



Then, solve the problem "Willa made 3 gallons of applesauce. She gave away 18 pints of the applesauce to her friends. How many pints of applesauce does she have remaining?" First, determine the total number of pints in 3 gallons, and then subtract the 18 pints she gave away. Since 1 gallon contains 8 pints, $3 \times 8 = 24$ shows that Willa made 24 pints of applesauce. Then, subtract the 18 pints to find that there are 6 pints of applesauce remaining because $24 - 18 = 6$.

Key Academic Terms:

measurement system, unit, conversion, multiplication, division, equivalent measurements, metric system, US Customary system

Additional Resources:

- Activity: Measurement: converting capacity task cards
- Video: Metric System
- Video: Metric System
- Video: Inches, feet, & yards
- Video: Ounces, pounds, and tons
- Video: Capacity

18a

Measurement
Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.
18. Identify volume as an attribute of solid figures, and measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised (non-standard) units. a. Pack a solid figure without gaps or overlaps using *n* unit cubes to demonstrate volume as *n* cubic units.

Guiding Questions with Connections to Mathematical Practices:

What is volume and what are examples of objects that do and do not have volume?

M.P.4. Model with mathematics. Know that volume represents the amount of space enclosed within a three-dimensional figure. For example, a cube has volume, but a square does not. Additionally, all three-dimensional figures have volume and all two-dimensional figures do not have volume.

- Ask students to determine if a given item has volume or not. For example, give students the list of items shown.
  - Basketball
  - Car
  - Cylinder
  - Projection of a photograph
  - Classroom

  The only item on this list that does not have volume is the projection of a photograph.

- Ask students to make lists of items with volume and items without volume. A possible student response is shown.

Has volume	Does not have volume
water bottle backpack baseball	triangle rectangle

What is a unit cube and how does it relate to volume?

M.P.4. Model with mathematics. Know that a unit cube is a cube with a side length of 1 unit and a volume of 1 cubic unit, and it can be used to represent and measure the volume of solid figures. For example, if a three-dimensional shape contains the same amount of space as a cube with a side length of 1 inch, then that shape has a volume of 1 cubic inch. Additionally, any unit of length can be used to make a unit cube. For example, a unit cube can have edge lengths of 1 inch, 1 centimeter, or 1 foot.

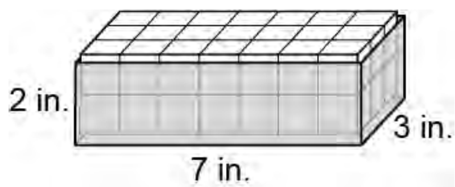
- Ask students to use a moldable material, like modeling clay, to create a unit cube of a certain measurement. For example, ask students to sculpt a cube and use a ruler to measure the edges, adjusting the size to be one-inch edge lengths or one-centimeter edge lengths, as needed.

- Ask students to explore how unit cubes are related to volume by submerging an object into water, measuring the water level, and then finding how many unit cubes are needed to make the water rise to the same level as the object. For example, a large eraser submerged in a container of water raises the level of the water by 1 centimeter. In the same container, it takes 9 cubes with edge lengths of 1 centimeter to make the same amount of water rise 1 centimeter, so the volume of the eraser is 9 cubic centimeters.

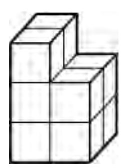
How can multiple cubes be used to measure the volume of a figure?

M.P.4. Model with mathematics. Know that if the amount of space enclosed within a three-dimensional figure is exactly the same as the amount of space within a set of cubes, then the volume of the figure can be expressed as the number of cubes used to fill the figure. For example, if a figure is packed with 12 cubes with side lengths of 1 centimeter with no gaps or overlaps, then the volume of the figure is 12 cubic centimeters. Additionally, objects can contain layers of unit cubes that may or may not have the same number of cubes in each layer.

- Ask students to find how many unit cubes fit in a given container using manipulatives, and then determine the volume of the container. For example, a box with dimensions 3 inches by 7 inches by 2 inches can be layered with 1-inch cubes with no gaps or overlaps by laying 3 rows of 7 in the bottom of the box and then adding an identical second layer of cubes on the top of the bottom layer. Since the total number of unit cubes that fit in the box is 42, the volume of the box is 42 cubic inches.

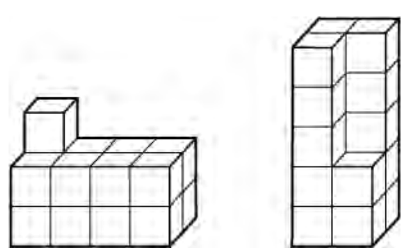


- Ask students to find the volume of a figure composed of unit cubes. For example, the figure shown has 4 unit cubes in the bottom layer, 4 unit cubes in the middle layer, and 2 unit cubes in the top layer.



The volume of the figure is 10 cubic units because $4 + 4 + 2 = 10$.

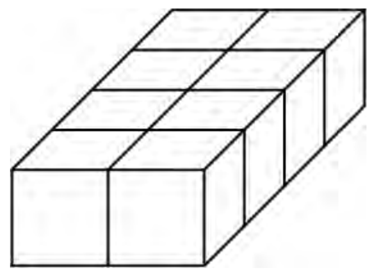
- Ask students to use unit cubes to build figures with a given volume. For example, ask students to make a figure with a volume of 17 cubic units using unit cube manipulatives. Some possible figures are shown.



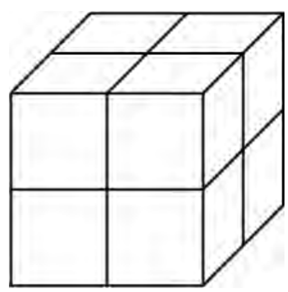
How can the volume of a right rectangular prism that is composed of identical cubes be determined?

M.P.6. Attend to precision. Know that when a right rectangular prism is composed of identical cubes with no gaps or overlaps, counting the total number of cubes produces a measurement of the volume of the prism. For example, if a figure is composed entirely of 24 cubes with edge lengths of 1 inch, then the volume of the figure is 24 cubic inches. Additionally, look for patterns to make counting the number of cubes easier. Further, cubes that have edge lengths that are not exactly one unit can also be used to find volume.

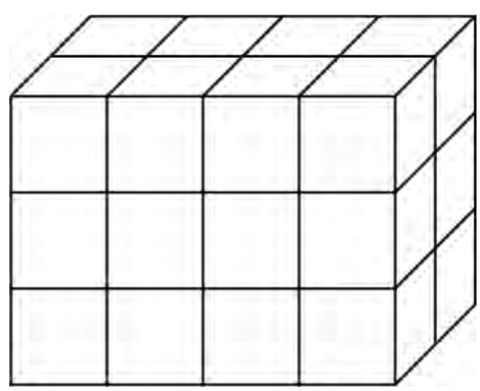
- Ask students to find the volume of a right rectangular prism given a diagram composed of unit cubes. For example, give students the following diagrams.



The volume is 8 cubic units.



The volume is also 8 cubic units.



The volume is 24 cubic units.

- Ask students to find the volume of a right rectangular prism using an improvised unit. For example, "Tobey has plastic blocks with edge lengths of 2 inches and a box that is 14 inches long, 10 inches wide, and 6 inches deep." Since the depth of Tobey's box is 6 inches, 3 layers of plastic blocks fit in the box. The length of 14 inches allows for 7 rows of blocks, and the width of 10 inches means 5 blocks fit in each row. So, 5×7 means each layer contains 35 blocks, and since 3 layers fit, 35×3 means a total of 105 plastic blocks fit in the box. Therefore, Tobey's box has a volume of 105 plastic blocks.

Key Academic Terms:

volume, space, cube, unit cube, three-dimensional, cubic unit, attribute, solid figure, cubic centimeters, cubic inches, cubic feet

Additional Resources:

- Video: Volume
- Activity: Box of clay
- Lesson: Volume of a composite solid
- Lessons: Grade 5 mathematics module 5

19a

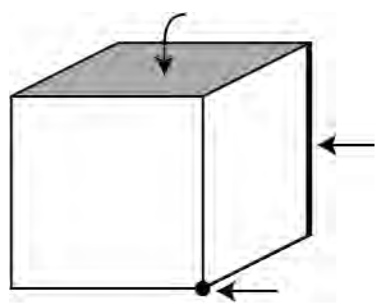
Measurement
Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.
19. Relate volume to the operations of multiplication and addition, and solve real-world and mathematical problems involving volume. a. Use the associative property of multiplication to find the volume of a right rectangular prism and relate it to packing the prism with unit cubes. Show that the volume can be determined by multiplying the three edge lengths or by multiplying the height by the area of the base.

Guiding Questions with Connections to Mathematical Practices:

What are the attributes of a right rectangular prism?

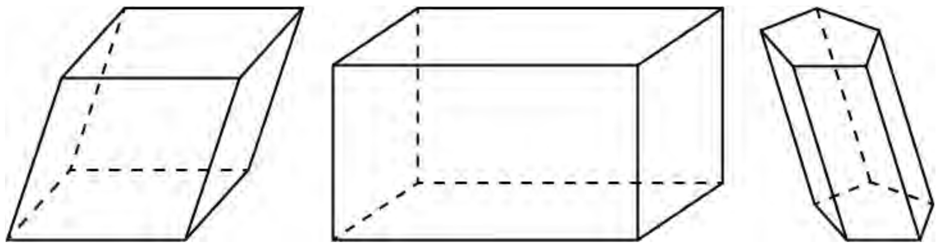
M.P.2. Reason abstractly and quantitatively. Know that a right rectangular prism is a solid object with 6 faces that are all rectangles. For example, a cube is a rectangular prism because the bases are congruent rectangles. Additionally, a right rectangular prism has 12 edges and 8 vertices. Further, right rectangular prisms have all right angles.

- Ask students to identify the vertices, edges, and faces of a solid figure. For example, ask students to identify the indicated parts of the figure shown.



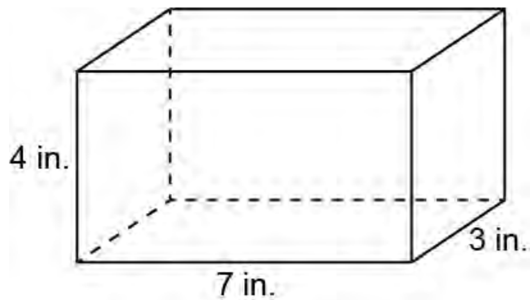
The shaded portion is a face of the figure because it is a flat polygon on the surface of the figure. The bold segment is an edge because it is a segment where two faces meet. The dot is a vertex because it is a point where multiple edges meet.

- Ask students to determine if a shape is a right rectangular prism or not. For example, give students the figures shown.

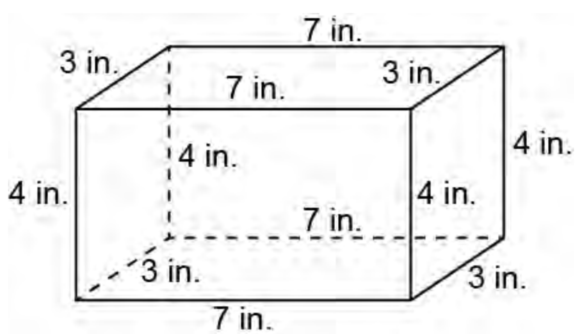


Look for figures with 6 rectangular faces and all right angles. Of the figures shown, only the figure in the middle is a right rectangular prism.

- Ask students to determine all the edge lengths of a right rectangular prism. For example, in the right rectangular prism shown, write in the remaining edge length units.



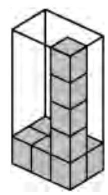
Since the front face shows a height of 4 inches, that means all the heights in the prism are 4 inches. The front face shows a length of 7 inches on the bottom edge, so the top edge of the front face is also 7 inches, as well as the top and bottom edges of the face opposite the front. The width is shown as 3 inches, so the remaining edge lengths in the prism are 3 inches.



How can the volume of a right rectangular prism that is packed with unit cubes be determined without counting the total number of cubes?

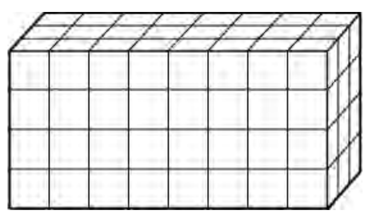
M.P.7. Look for and make use of structure. Observe that right rectangular prisms are composed of layers of unit cubes and connect the volume of the right rectangular prism to the area of a single layer multiplied by the total number of layers. For example, a right rectangular prism with a length of 3 inches, a width of 4 inches, and a height of 5 inches contains a total of 60 1-inch cubes, which is equivalent to $(3 \times 4) \times 5$. Additionally, the layers of unit cubes can be looked at from different viewpoints and the pattern of cubes will result in the same volume.

- Ask students to determine the number of unit cubes that fit in a right rectangular prism given a partially packed prism. For example, give students the diagram shown.

  The base of the prism fits 6 unit cubes, and it shows that 5 unit cubes can be stacked to find the number of layers that fit in the prism. To find the total number of unit cubes, multiply 6 by 5 and get 30. Therefore, the volume of the prism is 30 cubic units.

- Ask students to look for patterns in a rectangular prism packed with unit cubes and use the patterns to determine the volume. For example, give students the diagram shown.

  Note that the smallest dimensions are 3 units by 4 units, so the smallest face of the prism is made up of 12 unit cubes. Those 12 unit cubes appear across the prism 8 times. Therefore, the pattern of repeating 12 cubes 8 times means that the volume is 12×8, or 96 cubic units.

- Ask students to count the number of unit cubes in a rectangular prism without the use of a visual aid. For example, ask students to count the number of 1-inch cubes in a right rectangular prism that measures 3 inches wide, 5 inches long, and 2 inches high. The base of the prism is 3 inches by 5 inches. Because the length is 5 inches, 5 cubes can fit along the length of the base. Because the width is 3 inches, 3 rows of 5 cubes (15 cubes in total) fill the base. Because the height is 2 inches, 2 layers of cubes fit in the prism. The 2 layers of 15 cubes yield a total of 30 1-inch cubes.

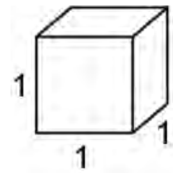
M.P.3. Construct viable arguments and critique the reasoning of others. Compare volumes of rectangular prisms to explore conservation of volume. For example, a rectangular prism with the dimensions of 2 units, 4 units, and 3 units will have the same volume as a rectangular prism with the dimensions of 1 unit, 2 units, and 12 units. Additionally, notice patterns in the dimensions of rectangular prisms to find volume.

- Ask students to construct rectangular prisms given a certain number of unit cubes. For example, give each student 12 unit cubes and ask them to use them all to construct a rectangular prism. Without any further guidelines, there will be prisms of different dimensions among the students. Possible prism dimensions using only 12 unit cubes are shown.

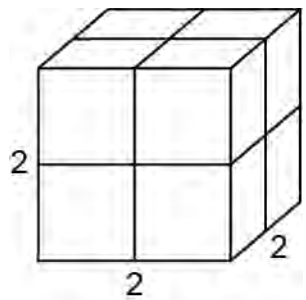
    - 1 unit by 1 unit by 12 units
    - 1 unit by 2 units by 6 units
    - 1 unit by 3 units by 4 units
    - 2 units by 2 units by 3 units

    Explain that each dimension can be considered the length, width, or height. Each prism has the exact same volume of 12 cubic units.

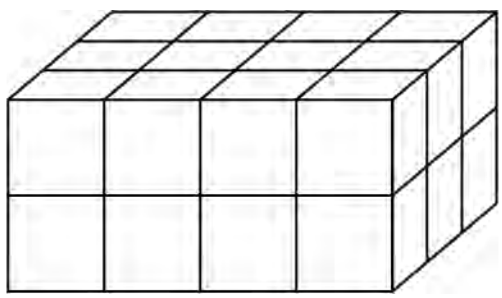
- Ask students to compare the volumes of rectangular prisms with dimensions that have been changed. For example, as a starting point, a cube with edge lengths of 1 unit has a volume of 1 cubic unit.



If the length of each edge is doubled, the new volume is NOT doubled. The new volume is $2 \times 2 \times 2 = 8$ cubic units, which is 8 times as great as the original volume.



As an additional example, give students the diagram shown.



Determine that the volume of the prism is $4 \times 3 \times 2 = 24$ cubic units. Then, ask the students to determine what the volume of the prism would be if all the dimensions of the prism (length, width, and height) were doubled. Students may guess that the volume is also doubled, which is incorrect. The length, width, and height each being multiplied by 2 means the volume is multiplied by 8. Discuss the reason why this is true. Since the original prism has dimensions of 4 units by 3 units by 2 units, the new prism has dimensions of 8 units by 6 units by 4 units. Therefore, the volume of the new prism is $8 \times 6 \times 4 = 192$ cubic units, which is 8 times as great as the volume of the original prism.

Key Academic Terms:

volume, unit cube, rectangular prism, base, face, length, width, height, congruent rectangles, equivalent, conservation of volume, layer, attribute, edge, vertex

Additional Resources:

- Video: Volume
- Activity: Using volume to understand the associative property of multiplication

19b

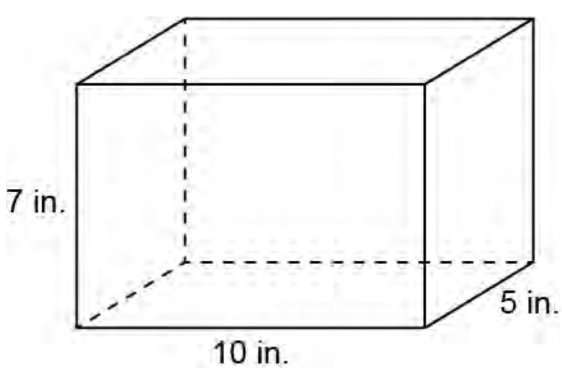
Measurement
Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.
19. Relate volume to the operations of multiplication and addition, and solve real-world and mathematical problems involving volume. b. Apply the formulas $V = l \times w \times h$ and $V = B \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real-world and mathematical problems.

Guiding Questions with Connections to Mathematical Practices:

What does each of the variables represent in the formulas $V = B \times h$ and $V = l \times w \times h$, and how do the formulas $V = B \times h$ and $V = l \times w \times h$ relate to each other?

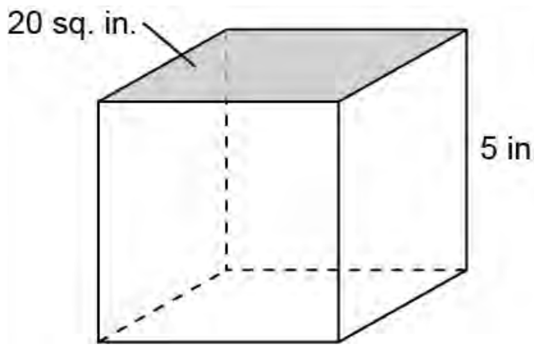
M.P.2. Reason abstractly and quantitatively. Define each variable in the formulas $V = B \times h$ and $V = l \times w \times h$ as Volume = Base × height and Volume = length × width × height. For example, if the edge lengths of a right rectangular prism are 2 inches, 4 inches, and 6 inches, then any two of those values can represent the base B (length × width), while the remaining value represents the height. Additionally, the formula $V = B \times h$ can be used to find the volume of any right prism where the area of the base is known. For example, if the area of the base of a right triangular prism is 24 square inches and the height is 5 inches, the volume is 120 cubic inches.

- Ask students to find the volume of a prism when the length, width, and height are given. For example, a rectangular prism with a length of 10 inches, a width of 5 inches, and a height of 7 inches is shown.



Find the volume of the given prism by using the formula $V = l \times w \times h$. The volume is 350 cubic inches because $10 \times 5 \times 7 = 350$.

- Ask students to find the volume of a prism where the area of the base and the height are given. For example, a right rectangular prism with a base of 20 square inches and a height of 5 inches is shown.

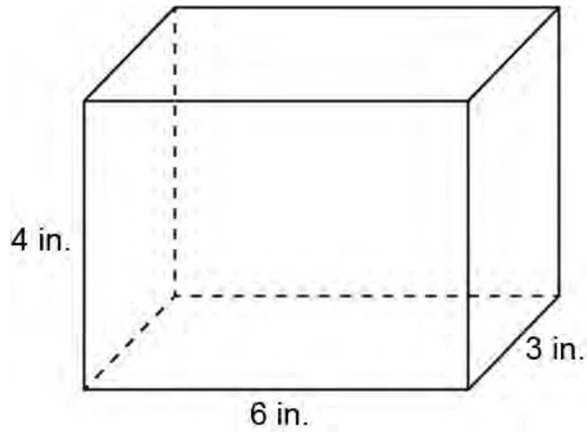


Find the volume of the given prism by using the formula $V = B \times h$. The volume is 100 cubic inches because $20 \times 5 = 100$.

Why does switching the length and width of a rectangular prism have no effect on the volume of the prism?

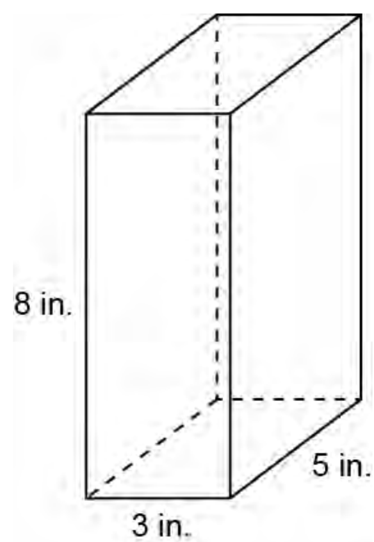
M.P.3. Construct viable arguments and critique the reasoning of others. Observe that the formula for calculating volume of a rectangular prism ($V = l \times w \times h$) only requires multiplication. Since multiplication is commutative, the order of multiplication can be rearranged without changing the product. For example, if a rectangular prism has a length of 5 centimeters, a width of 3 centimeters, and a height of 6 centimeters, the volume of the rectangular prism would be $5 \times 3 \times 6 = 15 \times 6 = 90$ cubic centimeters. If the length and width are switched, the calculation $3 \times 5 \times 6$ also equals $15 \times 6 = 90$ cubic centimeters. Additionally, by changing perspective, any of the 6 faces of a rectangular prism can be used as the base.

- Ask students to find the volume of a rectangular prism using the formula $V = l \times w \times h$. For example, use the dimensions of the rectangular prism shown.

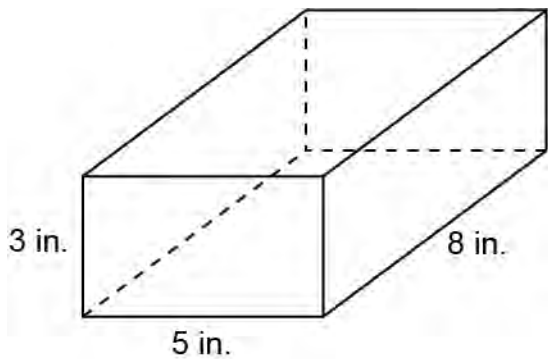


Find the volume by using 6 inches for the length, 3 inches for the width, and 4 inches for the height. The volume is 72 cubic inches because $6 \times 3 \times 4 = 72$. Then, have students switch the dimensions and multiply in a different order to discover that the volume is 72 cubic inches, regardless of which numbers are used for each dimension.

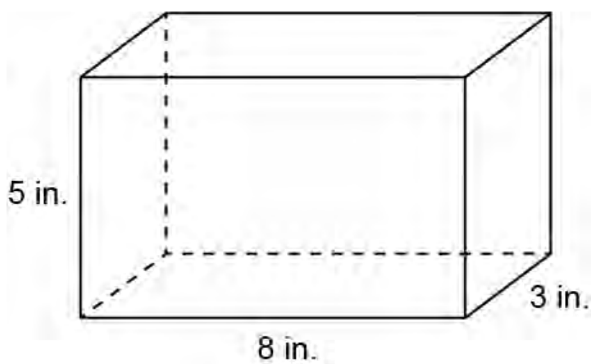
- Ask students to find the three possible base areas of a rectangular prism and the volume of the prism. For example, find the base areas of the rectangular prism shown with given edge lengths of 3 inches, 5 inches, and 8 inches.



Using the dimensions of 3 inches and 5 inches, students should find that the base area is 15 square inches. Then, recognizing that the height is 8 inches, students use the formula $V = B \times h$ to find the volume of 120 cubic inches because $15 \times 8 = 120$. Another view of the same rectangular prism is shown.



The base of the prism now has the dimensions of 5 inches and 8 inches, so the area of this base is 40 square inches. Recognizing that the height is now 3 inches, students use the formula $V = B \times h$ to find the volume of 120 cubic inches because $40 \times 3 = 120$. One more view of the same rectangular prism is shown.

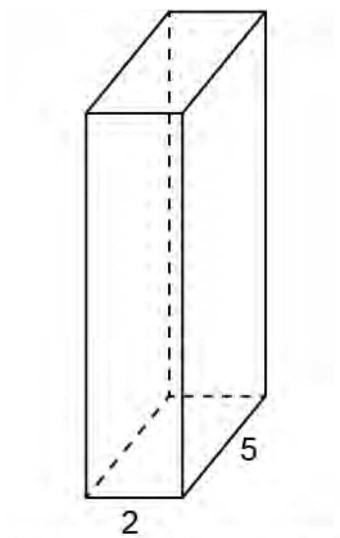


The base of the prism now has the dimensions of 8 inches and 3 inches, so the area of this base is 24 square inches. Recognizing that the height is now 5 inches, students use the formula $V = B \times h$ to find the volume of 120 cubic inches because $24 \times 5 = 120$. Therefore, the volume of the prism (which can be found using any of the three bases with the corresponding height measurement) is 120 cubic inches.

How can a right rectangular prism be constructed when the volume is known?

M.P.2. Reason abstractly and quantitatively. Determine possible values for the length, width, and height of a right rectangular prism by identifying the product of three values that equals the same value as the given volume. For example, if a right rectangular prism has a volume of 60 cubic centimeters, then 2 centimeters, 3 centimeters, and 10 centimeters are possible (but not exclusive) values for the length, width, and height of that prism because $2 \times 3 \times 10 = 60$. Additionally, if the volume and one edge length of a right rectangular prism are given, then the lengths of the other two dimensions of the prism would not be exclusive.

- Ask students to find the missing dimension of a right rectangular prism with a given volume and two edge lengths. For example, find the height of the right rectangular prism shown with edge lengths of 2 units and 5 units and a volume of 80 cubic units.



Volume = 80 cubic units

Find the missing dimension by using the known variables in the formula $V = l \times w \times h$. Using 80 for volume and 2 and 5 for length and width, the equation is $80 = 2 \times 5 \times h$. That means height, the missing dimension, is a number that results in 80 when multiplied by 2×5. Therefore, the height is 8 units because $80 \div 10 = 8$.

- Ask students to find possible lengths for two missing dimensions of a right rectangular prism if the volume and one edge length are known. For example, a rectangular prism has a volume of 90 cubic feet and a given side length of 3 feet. Since $90 \div 3 = 30$, possible answers are all factor pairs of 30, such as 1×30, 2×15, 3×10, and 5×6.

Key Academic Terms:

volume, rectangular prism, base, length, width, height, formula, edge, three-dimensional

Additional Resources:

- Lesson: Volume of a composite solid
- Video: Volume
- Video: Find the volume of a rectangular prism example!

19c

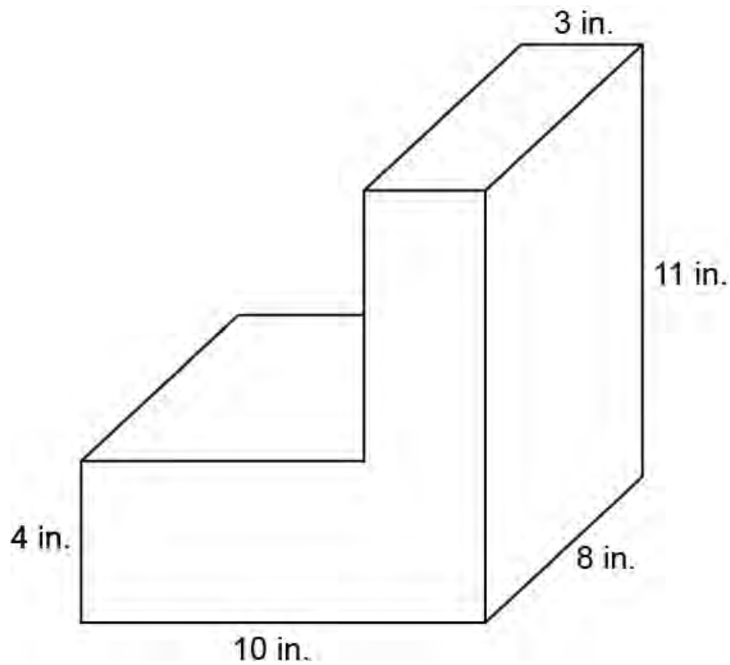
Measurement
Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.
19. Relate volume to the operations of multiplication and addition, and solve real-world and mathematical problems involving volume. c. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the two parts, applying this technique to solve real-world problems.

Guiding Questions with Connections to Mathematical Practices:

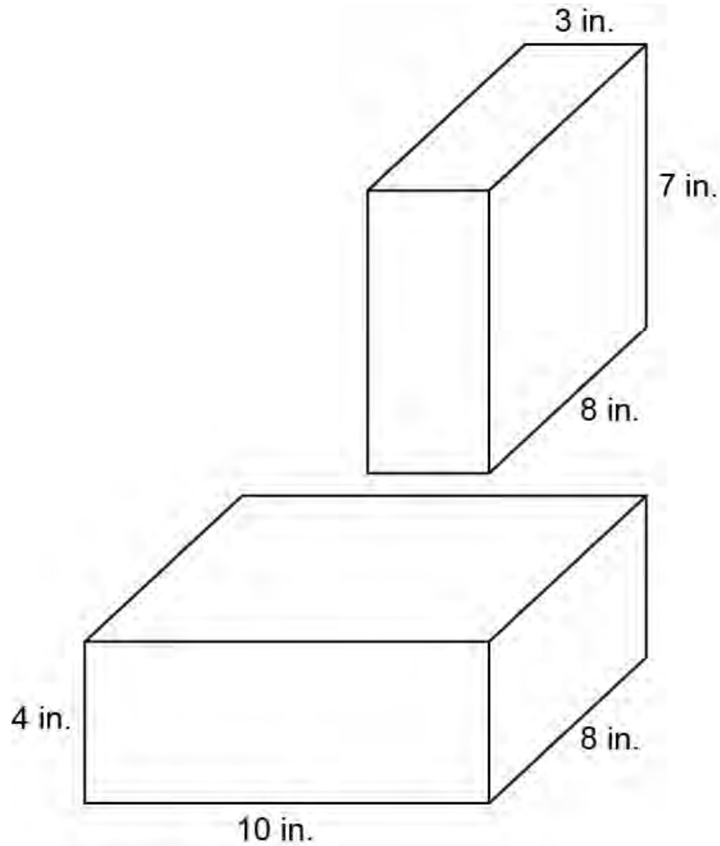
How can decomposition be used to determine the volume of solid figures?

M.P.4. Model with mathematics. Observe that if a three-dimensional solid figure can be decomposed into two or more right rectangular prisms, then the sum of the volumes of the right rectangular prisms is equivalent to the volume of the original solid figure. For example, if two prisms, one with dimensions of 16 inches, 8 inches, and 8 inches and one with dimensions of 16 inches, 8 inches, and 16 inches, are placed together to create a set of steps, then the total volume of the steps is 3,072 cubic inches because the sum of the volumes of the individual prisms, 1,024 and 2,048, equals 3,072. Additionally, decomposing a three-dimensional solid figure in more than one way still results in the same volume.

- Ask students to find the volume of a three-dimensional solid figure. For example, find the volume of the given figure.

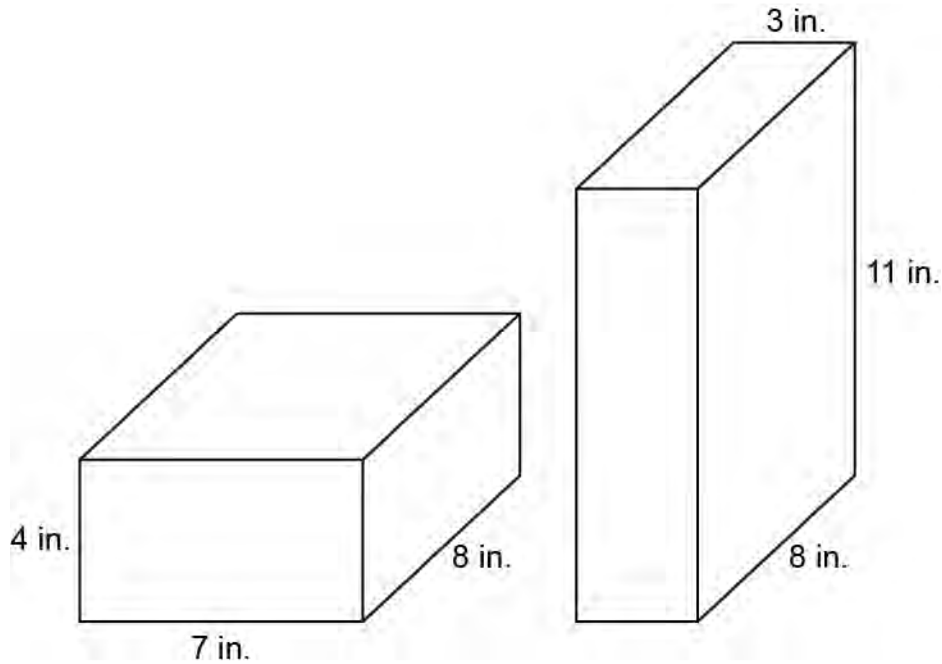


The figure can be decomposed into two rectangular prisms.



The volume of the bottom prism is 320 cubic inches, which is equal to $10 \times 8 \times 4$. The volume of the top prism is 168 cubic inches, which is equal to $3 \times 8 \times 7$. The total volume of the figure is 488 cubic inches because $320 + 168 = 488$.

Additionally, observe that there are often multiple ways that a given figure can be decomposed. Some students may choose to decompose the prism as shown.



The volume of the left prism is 224 cubic inches, which is equal to $7 \times 8 \times 4$. The volume of the right prism is 264 cubic inches, which is equal to $3 \times 8 \times 11$. The total volume of the figure is 488 cubic inches because $224 + 264 = 488$. Both decompositions result in the same volume.

- Ask students to find the volume of a figure made up of multiple rectangular prisms in a real-world context. For example, "A warehouse is currently in the shape of a right rectangular prism and is 50 feet long, 40 feet wide, and 25 feet high. An addition, which is also in the shape of a right rectangular prism, will be built on the side of the existing warehouse. The addition will be 20 feet long, 40 feet wide, and 25 feet high." To find the volume of the expanded warehouse, the volume of the original warehouse is added to the volume of the addition. Some possible student work is shown.

Original Warehouse: $50 \times 40 \times 25 = 50{,}000$

Addition: $20 \times 40 \times 25 = 20{,}000$

Total: $50{,}000 + 20{,}000 = 70{,}000$

The volume of the expanded warehouse will be 70,000 cubic feet.

Key Academic Terms:

volume, rectangular prism, composition, decomposition, additive, three-dimensional, sum, equivalent

Additional Resources:

- Lesson: Volume of a composite solid
- Tutorial: Volumes of composite figures

20

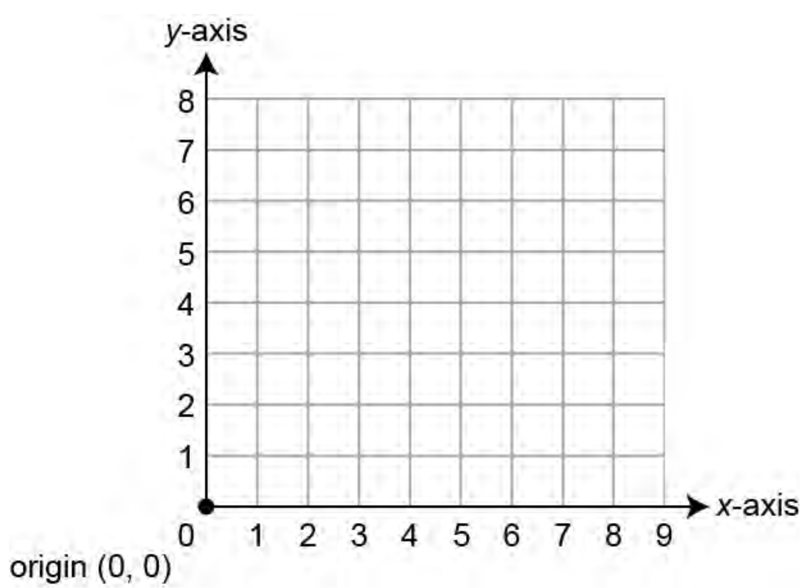
Geometry
Graph points on the coordinate plane to solve real-world and mathematical problems.
20. Graph points in the first quadrant of the coordinate plane, and interpret coordinate values of points to represent real-world and mathematical problems.

Guiding Questions with Connections to Mathematical Practices:

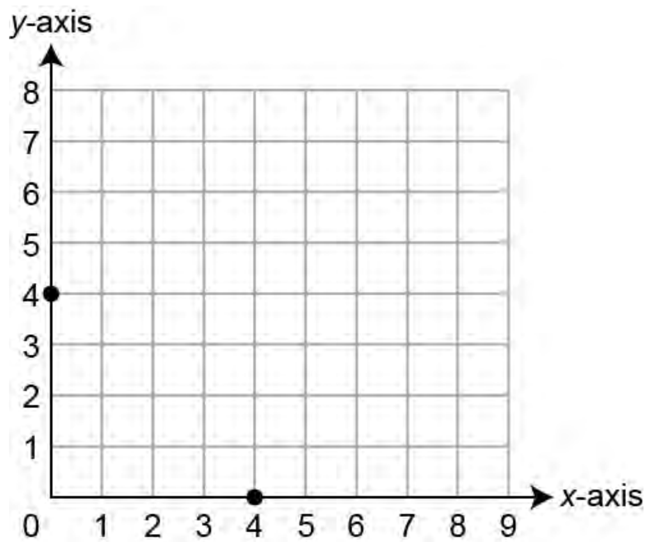
What does the origin of a coordinate system represent and how is it used?

M.P.5. Use appropriate tools strategically. Know that the origin of a coordinate system is the location where the horizontal *x*-axis intersects with the vertical *y*-axis, and that this location is used to determine the location of all other ordered pairs. For example, the ordered pair (0, 0) represents the origin, and all other ordered pairs refer to a horizontal and vertical distance from this location. Additionally, if moving zero units to the right of the origin and then zero units up, the resulting location is given the coordinates (0, 0).

- Give students a copy of the first quadrant. Ask students to draw a dot on the point where the horizontal axis intersects with the vertical axis, label that point "(0, 0)," and write the word "origin."



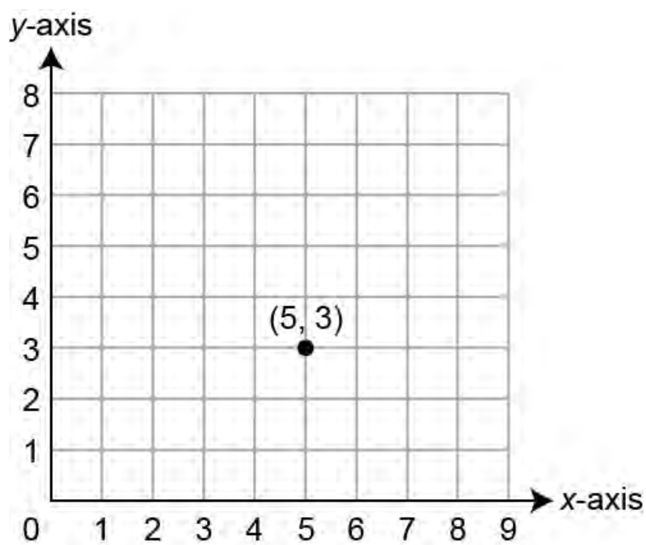
- Ask students to plot a point on the coordinate grid, given instructions for the location along the *x*- or *y*-axis. For example, ask students to plot a point at a location that is four units from the origin along the *x*-axis. Then, ask students to plot a point at a location that is four units from the origin along the *y*-axis. Ask students to discuss how to label the two points and how the labels indicate different locations. The order of coordinates is important to follow so that movement away from the origin results in the desired location.



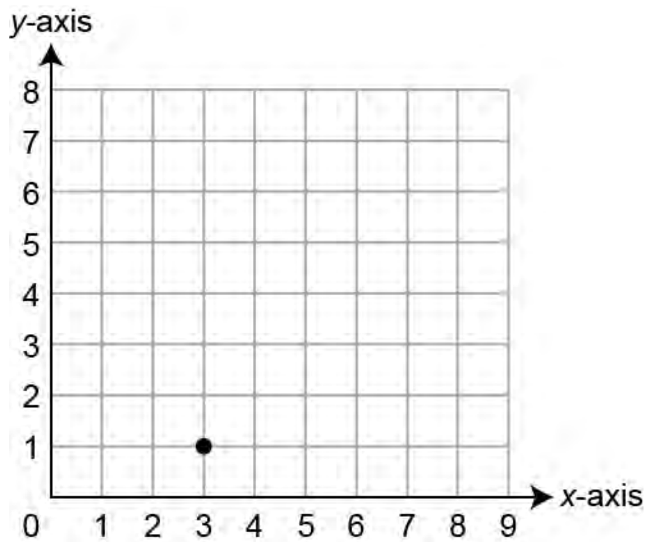
How is the location of an ordered pair in the first quadrant determined on a coordinate system?

M.P.5. Use appropriate tools strategically. Know that the first coordinate of an ordered pair indicates how far to move to the right from the origin while the second coordinate indicates how far to move above the origin. For example, the ordered pair (3, 2) indicates a location that is 3 units to the right of the origin and 2 units above that location on the *x*-axis. Additionally, the ordered pair (2, 3) indicates a different location that is 2 units to the right of the origin and 3 units above that location on the *x*-axis. The order of the numbers in the pair matters.

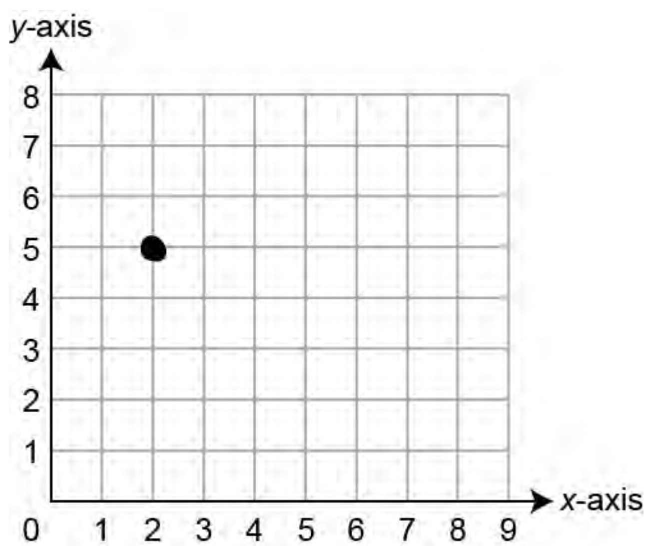
- Give students a point on the coordinate plane and ask them to identify the coordinate pair for this location by starting at the origin and moving right along the *x*-axis until under the point. Notice the label on the *x*-axis, as this corresponds to the first number in the coordinate pair. Then, move up from the location on the *x*-axis to the given point above it, counting units moved to determine the second number in the coordinate pair. For example, given the unlabeled point at (5, 3), know that the first number in the ordered pair, the *x*-coordinate, corresponds to the five units traveled along the *x*-axis, and know that the number 3 is the *y*-coordinate, the number of units traveled up.



- Ask students to explain and demonstrate that a point placed on the coordinate plane has an "over and then up" component and that the order of the numbers in the coordinate pair matters. For example, the components of the point (3, 1) provide directions on how to arrive at that point. If the numbers are not matched with the correct directions, the desired location will not be reached.



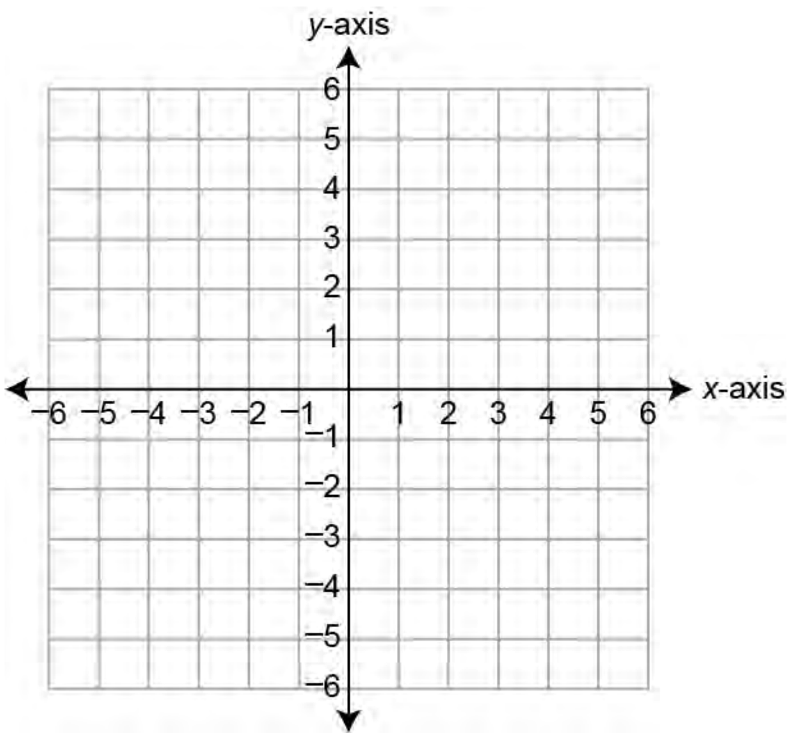
- Ask students to plot a point on the plane, given its coordinates. For example, if asked to plot (2, 5), students should start at the origin and move to the right along the x-axis two units, then travel up from that point five units and place a dot to label that as point (2, 5).



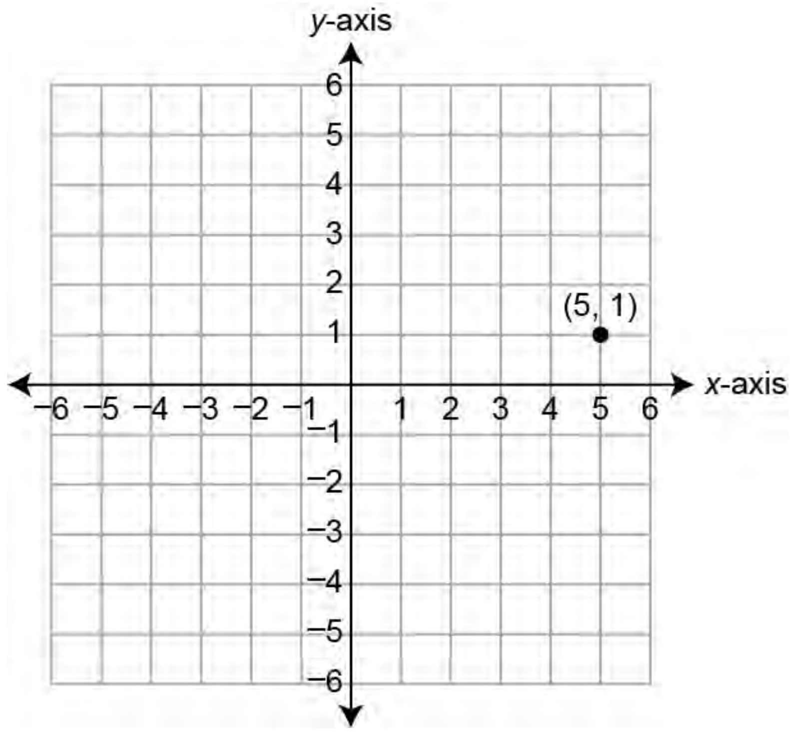
What is the first quadrant?

M.P.4. Model with mathematics. Know that the intersection of the horizontal *x*-axis and the vertical *y*-axis creates four distinct sections called quadrants, and that the first quadrant refers to the section that is above the *x*-axis and to the right of the *y*-axis. For example, the ordered pair (3, 2) is in the first quadrant because it is 3 units to the right of the *y*-axis and 2 units above the *x*-axis. Additionally, the points (5, 0) and (0, 1) are examples of points on the *x*- and *y*-axes, respectively, and are not in the first quadrant.

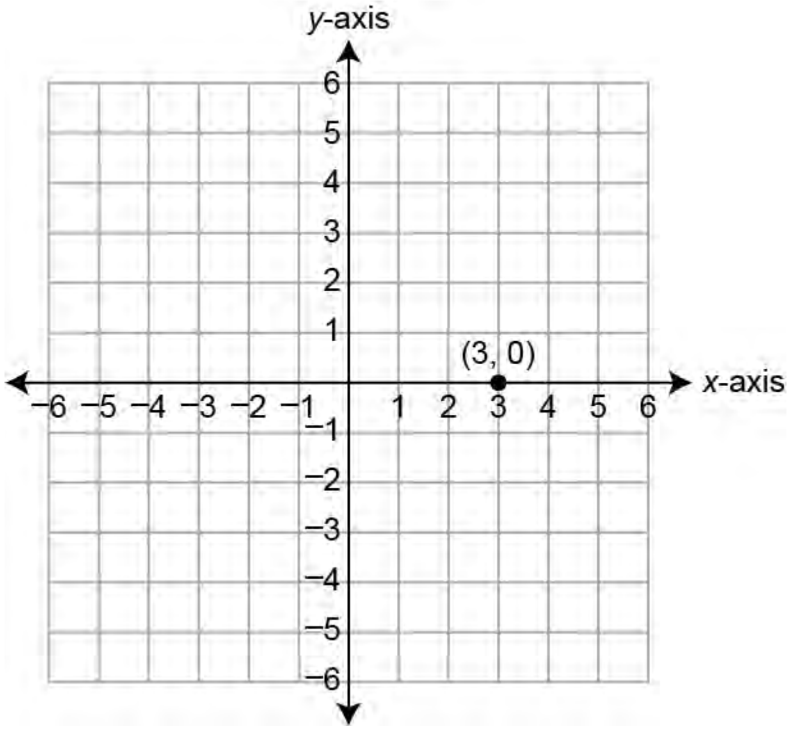
- Ask students to identify and place a point in the first quadrant. For example, when given the definition of the first quadrant, students may place a point anywhere that is both above the *x*-axis and to the right of the *y*-axis.



- Ask students to identify whether a point is in or is not in the first quadrant. For example, when shown the point (5, 1), identify the point as being in the first quadrant because it is both above the *x*-axis and to the right of the *y*-axis.



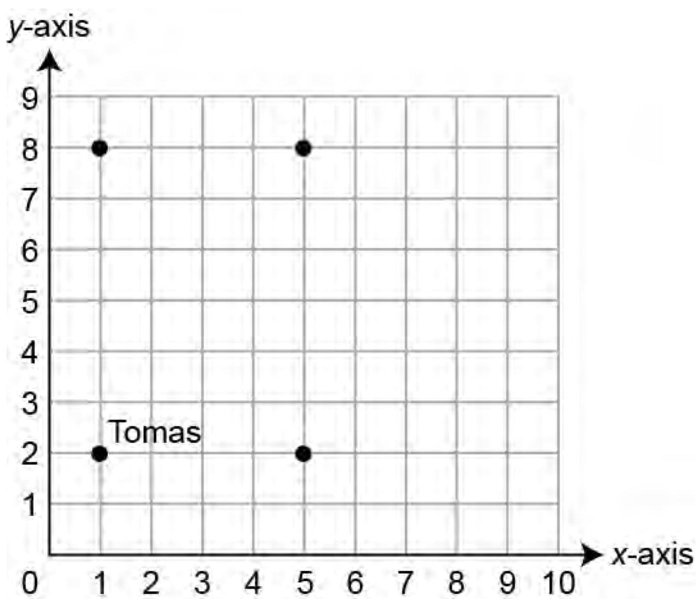
- Ask students to explain why a point on the *x*- or *y*-axis is not in the first quadrant. For example, when shown the point (3, 0), students identify the location as being ON the *x*-axis, not ABOVE it, recalling the definition of the first quadrant.



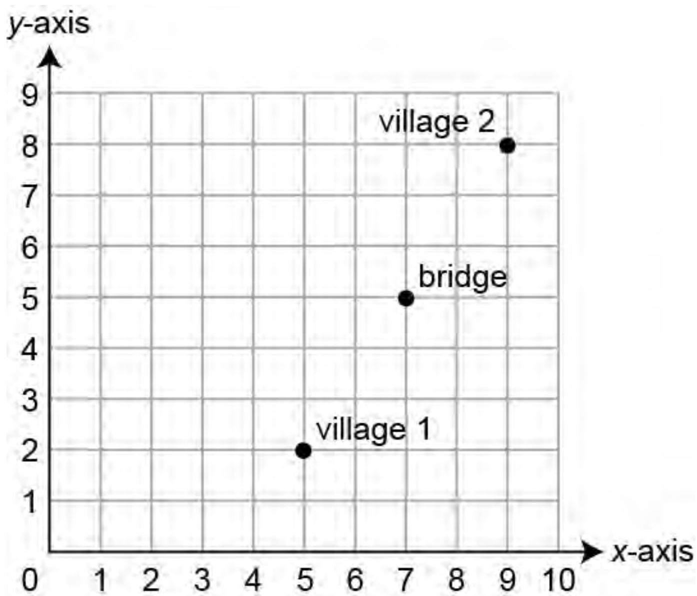
How can the first quadrant of the coordinate plane be used to represent real-world locations and situations?

M.P.4. Model with mathematics. Demonstrate that the coordinate plane can serve as a map that shows locations and the distances between locations. For example, if the library and the pharmacy are 2 miles apart on the same street, then the ordered pairs (2, 2) and (4, 2) could be used to represent these locations on a coordinate plane where 1 unit represents 1 mile, because (2, 2) and (4, 2) are exactly two units apart from each other. Additionally, place locations on the map and state or interpret distances using a coordinate plane.

- Ask students to map points on the coordinate plane and determine distances between them. For example, "The first quadrant represents the map of a city, with each unit a block; Tomas lives at (1, 2), and Tomas's friends live at (1, 8), (5, 8), and (5, 2)." Students should determine that the shortest distance Tomas can walk to reach all three friends and return home to (1, 2) is 20 blocks.



- Ask students to locate points on a coordinate plane in relation to other points already mapped. For example, "In a computer game, a programmer places villages on a coordinate plane at points (5, 2) and (9, 8)." Students determine that the location of a bridge that is exactly halfway between the villages is (7, 5).



Key Academic Terms:

coordinate system, coordinate, *x*-axis, *y*-axis, origin, ordered pair, plane, horizontal, vertical, coordinate plane, first quadrant

Additional Resources:

- Video: Coordinate plane Quadrant 1
- Lesson: Understand points on the coordinate plane
- Activity: Meerkat coordinate plane task

21

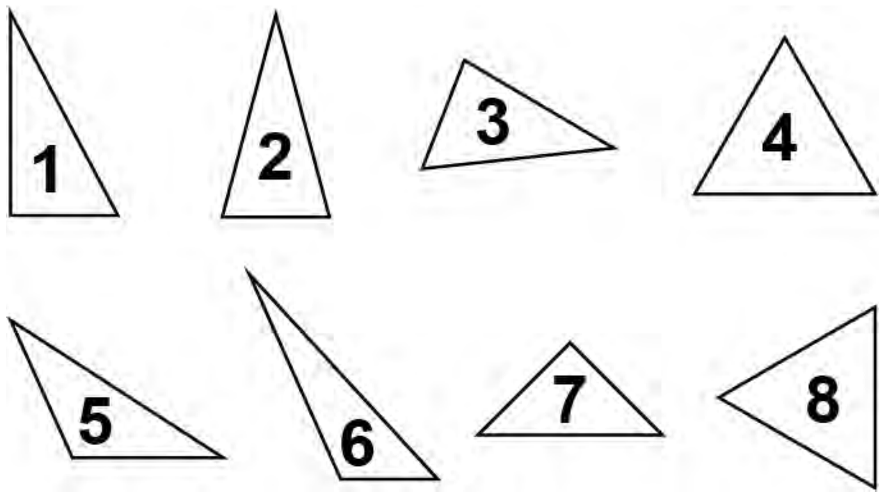
Geometry
Classify two-dimensional figures into categories based on their properties.
21. Classify triangles according to side length (isosceles, equilateral, scalene) and angle measure (acute, obtuse, right, equiangular).

Guiding Questions with Connections to Mathematical Practices:

What characteristics make a triangle isosceles, equilateral, or scalene, and what characteristics make a triangle acute, obtuse, right, or equiangular?

M.P.6 Attend to precision. Know that an equilateral triangle has 3 congruent sides, an isosceles triangle has 2 congruent sides, and a scalene triangle has no congruent sides, while an acute triangle has 3 acute angles, an obtuse triangle has 1 obtuse angle, a right triangle has 1 right angle, and an equiangular triangle has 3 angles with equal measure. For example, a triangle with side lengths of 4 centimeters, 4 centimeters, and 4 centimeters is equilateral; a triangle with side lengths of 4 centimeters, 4 centimeters, and 5 centimeters is isosceles; and a triangle with side lengths of 4 centimeters, 5 centimeters, and 6 centimeters is scalene. A triangle with angle measures of 50°, 60°, and 70° is acute; a triangle with angle measures of 35°, 45°, and 100° is obtuse; a triangle with angle measures of 40°, 50°, and 90° is right; and a triangle with angle measures of 60°, 60°, and 60° is equiangular. Additionally, classify a single triangle according to both its side lengths and its angle measures (e.g., acute isosceles).

- Present students with a collection of various triangles that are numbered. Then ask them to write down the numbers that correspond to triangles with different characteristics. For example, provide students with the following triangles numbered 1–8 as shown.



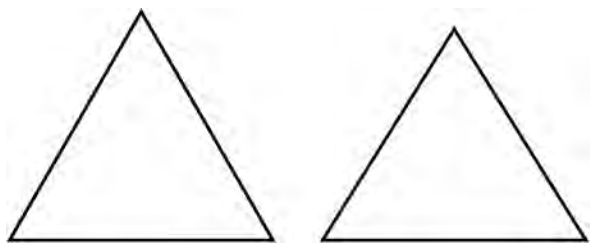
Then ask students to identify which triangles are isosceles and which triangles are acute. In this case, the isosceles triangles are numbers 2, 5, and 7, while the acute triangles are numbers 2, 3, 4, and 8. Continue the activity by asking students to identify triangles that are equilateral, scalene, right, and obtuse.

- Provide students with some characteristics of a triangle. Ask them to classify the triangle in two different ways. For example, tell students that a triangle has an angle that measures 90° and has side lengths of 3 centimeters, 4 centimeters, and 5 centimeters. In this case, students can classify the triangle as both scalene and right. For another example, tell students that a triangle has angles that measure 60°, 60°, and 60° and side lengths that measure 6 centimeters, 6 centimeters, and 6 centimeters. In this case, students can classify the triangle as equilateral, equiangular, and acute.

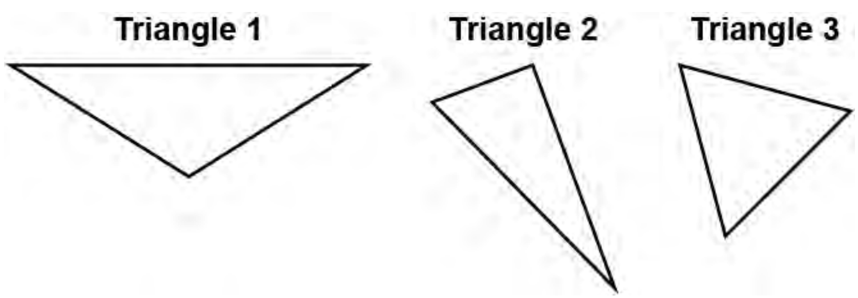
What tools can be used to help classify a triangle according to angle measure or side length when visual inspection is inadequate?

M.P.5 Use appropriate tools strategically. Use a protractor to measure one or more angles in a triangle to determine whether it should be classified as acute, obtuse, right, or equiangular. For example, if it is unclear visually whether the largest angle of a triangle is 90°, then a protractor can be used to confirm its measurement and conclude whether the triangle is acute, obtuse, or right. Additionally, use a ruler to measure the side lengths of a triangle to determine whether it should be classified as isosceles, equilateral, or scalene.

- Provide students with two or more acute triangles. Ask them to measure all the angles using a protractor to determine whether each triangle is equiangular. For example, students could measure the angles of the triangles shown below to determine that the first triangle is equiangular while the second triangle is not.



- Provide students with three different triangles. Ask them to measure the side lengths to determine whether each is equilateral, isosceles, or scalene. Then ask students to measure the angles to determine whether each is acute, obtuse, right, or equiangular. For example, in the triangles shown below, Triangle 1 is isosceles and obtuse, Triangle 2 is scalene and right, and Triangle 3 is equilateral and equiangular.



Key Academic Terms:

isosceles, equilateral, scalene, acute, obtuse, right, equiangular, degree, protractor

Additional Resources:

- Video: Types of triangles
- Activity: Polygraph: Triangles

22

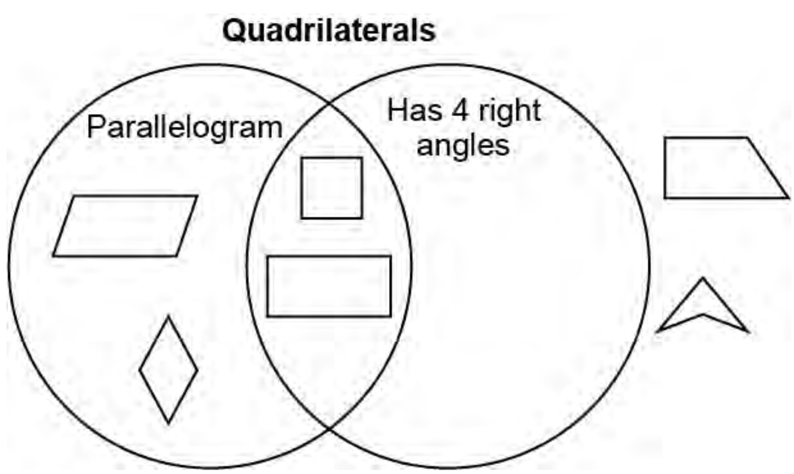
Geometry
Classify two-dimensional figures into categories based on their properties.
22. Classify quadrilaterals in a hierarchy based on properties.

Guiding Questions with Connections to Mathematical Practices:

How is a hierarchy of figures created based on properties?

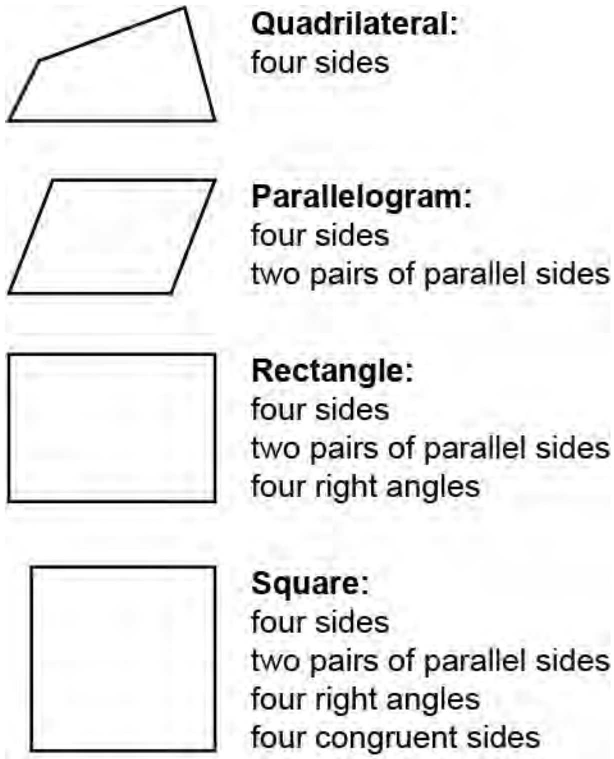
M.P.6. Attend to precision. Know that a hierarchy of figures is created by identifying and distinguishing properties that are more general from those that are more specific and by making connections between and within categories of figures. For example, a quadrilateral is a figure with the general property of having 4 sides, while a parallelogram is a specific type of quadrilateral that has two pairs of opposite sides that are both parallel and congruent. Based on this hierarchy, all parallelograms are quadrilaterals, but not all quadrilaterals are parallelograms. Additionally, all figures in a hierarchy have at least one general attribute in common, but they are differentiated by less general attributes in each step down the hierarchy.

- Give students a Venn diagram and ask them to place various shapes in the appropriate region of the Venn diagram. For example, the following Venn diagram shows two possible subcategories with several shapes that have been placed by students.



Discuss with students the fact that there are no shapes in the right portion of the Venn diagram because all quadrilaterals that have four right angles are also parallelograms.

- Ask students to classify figures in a hierarchy based on the properties of the figures. For example, students identify the properties of a square, a rectangle, a parallelogram, and a quadrilateral. Then, order the figures top to bottom such that all attributes of a figure are also attributes of the figures below it.



Key Academic Terms:

two-dimensional, properties

Additional Resources:

- Video: [Camp Quadrilaterals]
- Activity: [What do these shapes have in common?]

23

Geometry
Classify two-dimensional figures into categories based on their properties.
23. Explain that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. *Example: All rectangles have four right angles, and squares have four right angles, so squares are rectangles.*

Guiding Questions with Connections to Mathematical Practices:

How are attributes used to categorize and subcategorize two-dimensional figures?

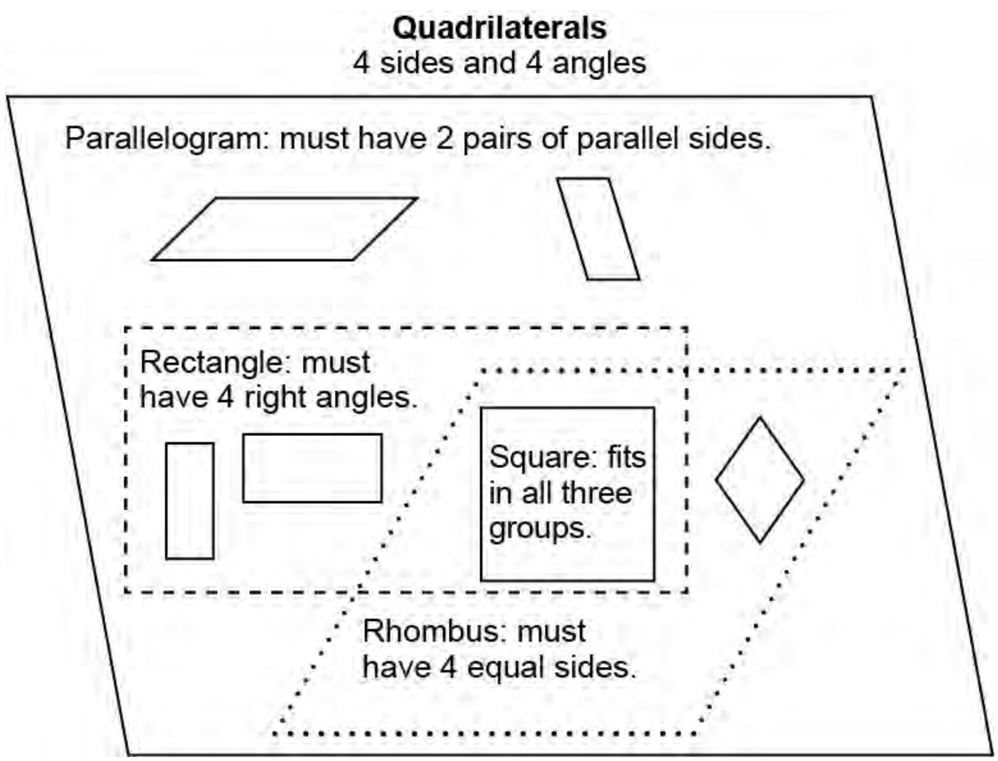
M.P.6. Attend to precision. Identify attributes such as number of sides, side lengths, angles, and presence of parallel or perpendicular lines to categorize and subcategorize two-dimensional figures. Use these attributes to make connections between categories and subcategories. For example, all rectangles, squares, and rhombuses have opposite sides that are parallel and congruent. As such, all rectangles, squares, and rhombuses are parallelograms. Additionally, observe that attributes belonging to a subcategory do not necessarily belong to the larger category.

- Ask students to list characteristics that define a shape as well as characteristics that are always true of a shape. Then, ask them to draw examples to illustrate this. For example, a rectangle is defined as a quadrilateral with exactly four right angles. However, there are several characteristics that all rectangles share beyond this. Some characteristics include:
  - Two pairs of parallel sides
  - Two pairs of equal sides
  - Diagonals that are the same length

  Ask students to draw several quadrilaterals that have four right angles. Then, observe that all the figures drawn will have the other characteristics listed, even when students did not intentionally try to include those characteristics.

- Ask students to identify a category of shapes as well as a subcategory. Ask students to list characteristics of the larger category and then observe that those characteristics are also true of the subcategory. For example, students identify that right triangles are a subcategory of triangles and then list properties of triangles. Any properties of triangles that students list are also true of the subcategory of right triangles. Some characteristics that students may observe are that triangles have three angles as well as three sides and that all triangles have at least two acute angles. Because right triangles are a subcategory of triangles, all those characteristics are true of right triangles as well. Discuss with students the fact that the converse does not hold true. While all properties of triangles are also true of right triangles, not all properties of right triangles are also true of triangles in general.

- After explaining that a particular class of shapes has a stated property, ask students to draw conclusions about which other shapes must also have that property based on shape categories and subcategories. For example, explain to students that the diagonals of a rhombus intersect at 90-degree angles and then ask students to draw conclusions about other shapes.



As shown in the diagram, squares are a subcategory of rhombuses. Therefore, the diagonals of a square also intersect at 90-degree angles. However, because rectangles are not a subcategory of rhombuses, the diagonals of a rectangle do not necessarily intersect at 90-degree angles. As a further example, explain that the opposite angles of a parallelogram have the same degree measure. Therefore, because squares, rectangles, and rhombuses are all subcategories of parallelograms, all squares, rectangles, and rhombuses also have this property.

Key Academic Terms:

attribute, category, subcategory, two-dimensional, figure, right angle, parallel, perpendicular

Additional Resources:

- Activity: Polygon task cards
- Video: Types of triangles
- Video: Polygons song
- Video: Camp Quadrilaterals

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February 15, 2022