ACAP
Alabama Comprehensive Assessment Program
SUMMATIVE

Grade 7
Mathematics

Alabama Educator Instructional Supports

Alabama Course of Study Standards

DATA RECOGNITION DRC CORPORATION

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Introduction

The *Alabama Instructional Supports: Mathematics* is a companion to the 2019 *Alabama Course of Study: Mathematics* for Grades K–12. Instructional supports are foundational tools that educators may use to help students become independent learners as they build toward mastery of the *Alabama Course of Study* content standards. **Instructional supports are designed to help educators engage their students in exploring, explaining, and expanding their understanding of the content standards.**

The content standards contained within the course of study may be accessed on the Alabama State Department of Education (ALSDE) website: https://www.alabamaachieves.org/. When examining these instructional supports, educators are reminded that content standards indicate minimum content—what all students should know and be able to do by the end of each grade level or course. Local school systems may have additional instructional or achievement expectations and may provide instructional guidelines that address content sequence, review, and remediation.

The instructional supports are organized by standard. Each standard's instructional support includes a statement of the content standard, guiding questions with connections to mathematical practices, key academic terms, and additional resources.

Content Standards

The content standards are the statements from the 2019 *Alabama Course of Study: Mathematics* that define what all students should know and be able to do at the conclusion of a given grade level or course. Content standards contain minimum required content and complete the phrase "Students will ______."

Guiding Questions with Connections to Mathematical Practices

Guiding questions are designed to create a framework for the given standards and to engage students in exploring, explaining, and expanding their understanding of the content standards provided in the 2019 *Alabama Course of Study: Mathematics*. Therefore, each guiding question is written to help educators convey important concepts within the standard. By utilizing guiding questions, educators are engaging students in investigating, analyzing, and demonstrating knowledge of the underlying concepts reflected in the standard. An emphasis is placed on the integration of the eight Student Mathematical Practices.

The Student Mathematical Practices describe the varieties of expertise that mathematics educators at all levels should seek to develop in their students. They are based on the National Council of Teachers of Mathematics process standards and the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up: Helping Children Learn Mathematics*.

The Student Mathematical Practices are the same for all grade levels and are listed below.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Each guiding question includes a representative set of sample activities and examples that can be used in the classroom. The set of activities and examples is not intended to include all the activities and examples that would be relevant to the standard.

Key Academic Terms

These academic terms are derived from the standards and are to be incorporated into instruction by the educator and used by the students.

Additional Resources

Additional resources are included that are aligned to the standard and may provide additional instructional support to help students build toward mastery of the designated standard. Please note that while every effort has been made to ensure all hyperlinks are working at the time of publication, web-based resources are impermanent and may be deleted, moved, or archived by the information owners at any time and without notice. Registration is not required to access the materials aligned to the specified standard. Some resources offer access to additional materials by asking educators to complete a registration. While the resources are publicly available, some websites may be blocked due to Internet restrictions put in place by a facility. Each facility's technology coordinator can assist educators in accessing any blocked content. Sites that use Adobe Flash may be difficult to access after December 31, 2020, unless users download additional programs that allow them to open SWF files outside their browsers.

Printing This Document

It is possible to use this entire document without printing it. However, if you would like to print this document, you do not have to print every page. First, identify the page ranges of the standards or domains that you would like to print. Then, in the print pop-up command screen, indicate which pages you would like to print.

1

Proportional Reasoning
Analyze proportional relationships and use them to solve real-world and mathematical problems.
1. Calculate unit rates of length, area, and other quantities measured in like or different units that include ratios or fractions.

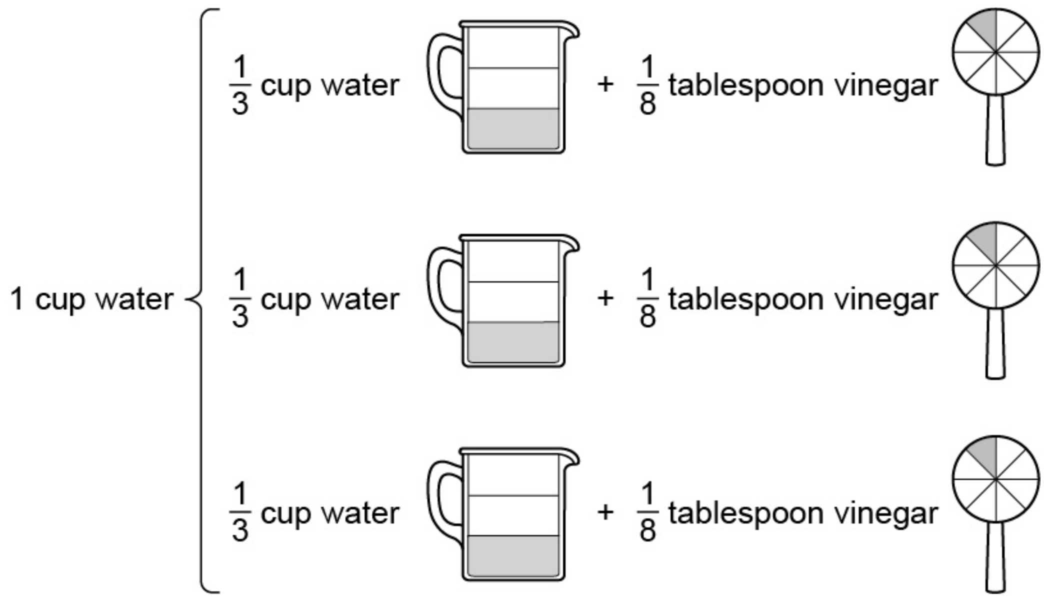
Guiding Questions with Connections to Mathematical Practices:

How can unit rate problems involving fractions be solved in a variety of ways?

M.P.1. Make sense of problems and persevere in solving them. Use a variety of methods to solve unit rate problems involving fractions. For example, use unit rate and proportional thinking to solve the problem "Josie makes a fruit drink using $\frac{2}{3}$ cup of orange juice for every $\frac{1}{4}$ cup of pineapple juice. How many cups of orange juice will Josie need if she uses 2 cups of pineapple juice?" by finding the amount of orange juice needed for 1 cup of pineapple juice using the proportion $\frac{\frac{2}{3}\text{ cup orange juice}}{\frac{1}{4}\text{ cup pineapple juice}} = \frac{x\text{ cups orange juice}}{1\text{ cup pineapple juice}}$. Use proportional reasoning to multiply both $\frac{2}{3}$ and $\frac{1}{4}$ by 4 to get the unit rate of $\frac{8}{3}$ cups of orange juice. Then, multiply $\frac{8}{3}$ by 2 to find the amount of orange juice needed for 2 cups of pineapple juice; Josie needs $5\frac{1}{3}$ cups of orange juice. Additionally, visual models can be used to represent and solve unit rate problems involving fractions.

- Ask students to solve unit rate problems with proportional thinking. For example, give students the situation “Amir can ride his bike $\frac{1}{2}$ mile in $\frac{1}{10}$ of an hour. How many miles can Amir ride his bike in 2 hours?” Use proportional thinking to determine Amir’s unit rate. The rate at which Amir rides his bike is $\frac{\frac{1}{2}\text{ mile}}{\frac{1}{10}\text{ hour}}$. To determine the unit rate, the denominator should be 1 hour, so multiply $\frac{1}{10}$ of an hour by 10 and $\frac{1}{2}$ mile by 10 to find that the unit rate is $\frac{5\text{ miles}}{1\text{ hour}}$, or 5 miles per hour. Then, use proportional thinking again to determine that since Amir’s unit rate is 5 miles per hour, Amir can ride his bike 10 miles in 2 hours.

- Ask students to solve unit rate problems using a visual model. For example, give students the situation "Jephte uses $\frac{1}{8}$ tablespoon of vinegar in $\frac{1}{3}$ cup of water to make a cleaning solution. How many tablespoons of vinegar will Jephte need to make a cleaning solution that has 4 cups of water?" A sample student visual model and response are shown.



There are three $\frac{1}{8}$ tablespoons of vinegar for each 1 cup of water. So there are $\frac{3}{8}$ tablespoons of vinegar for each 1 cup water.

$$\frac{3}{8} \times 4 = \frac{12}{8} = 1\frac{1}{2}$$

There are $1\frac{1}{2}$ tablespoons of vinegar in a cleaning solution with 4 cups of water.

Key Academic Terms:

unit rate, ratio, unit, complex fraction, per, for every

Additional Resources:

- Activities: Unit rate
- Activities: Ratios & proportions: blueprinting our classroom
- Activity: Track practice

2a

Proportional Reasoning
Analyze proportional relationships and use them to solve real-world and mathematical problems.
2. Represent a relationship between two quantities and determine whether the two quantities are related proportionally. a. Use equivalent ratios displayed in a table or in a graph of the relationship in the coordinate plane to determine whether a relationship between two quantities is proportional.

Guiding Questions with Connections to Mathematical Practices:

What makes a relationship proportional?

M.P.6. Attend to precision. Define proportional relationships and determine their characteristics. For example, identify that proportional relationships contain pairs of numbers that are all equivalent ratios. Additionally, this common ratio can be used to identify other pairs of numbers in the relationship.

- Ask students to explain how to determine whether a relationship is a proportional relationship. For example, a small bag of flour weighs 2 pounds and costs $2, a medium bag of flour weighs 5 pounds and costs $4, and the largest bag of flour weighs 10 pounds and costs $7. In order to determine whether the weight of the flour and the cost of the flour are in a proportional relationship, calculate the ratio between the two quantities for each bag size. The ratio can be calculated in either order as long as the order remains consistent. The ratios of pounds to dollars for the three bags are 2:2, 5:4, and 10:7. Because the ratios are not equivalent, the relationship between weight and cost is not proportional.

- Ask students to find additional pairs of values in a proportional relationship. For example, a company charges $8 to rent a kayak for 1 hour, $16 to rent a kayak for 2 hours, and $24 to rent a kayak for 3 hours. The relationship between rental cost and time is proportional because the ratios between cost and time are 8:1, 16:2, and 24:3, all of which are equivalent to 8. Another pair of values in the proportional relationship is $40 and 5 hours, because the ratio 40:5 is also equivalent to 8.

M.P.3. Construct viable arguments and critique the reasoning of others. Construct examples and non-examples of proportional relationships. For example, a situation comparing the ages of two people over time does not show a proportional relationship, whereas a situation involving mixed quantities of paint where the shade of paint remains the same regardless of the quantity of paint shows a proportional relationship. Additionally, use ratios in each situation to explain the example.

- Ask students to give examples of situations that show proportional relationships.

  Jasmine makes homemade bird feed. She uses 2 cups of oats and $\frac{3}{4}$ cup of sunflower seeds to make 1 batch. To make a week's worth of bird feed, Jasmine needs to make 7 batches. She multiplies both the amount of oats and the amount of sunflower seeds by 7. There are 14 cups of oats and $5\frac{1}{4}$ cups of sunflower seeds in 7 batches of bird feed. The amount of oats and the amount of sunflower seeds are in a proportional relationship because the ratio of oats to sunflower seeds is the same for 1 batch as it is for 7 batches.

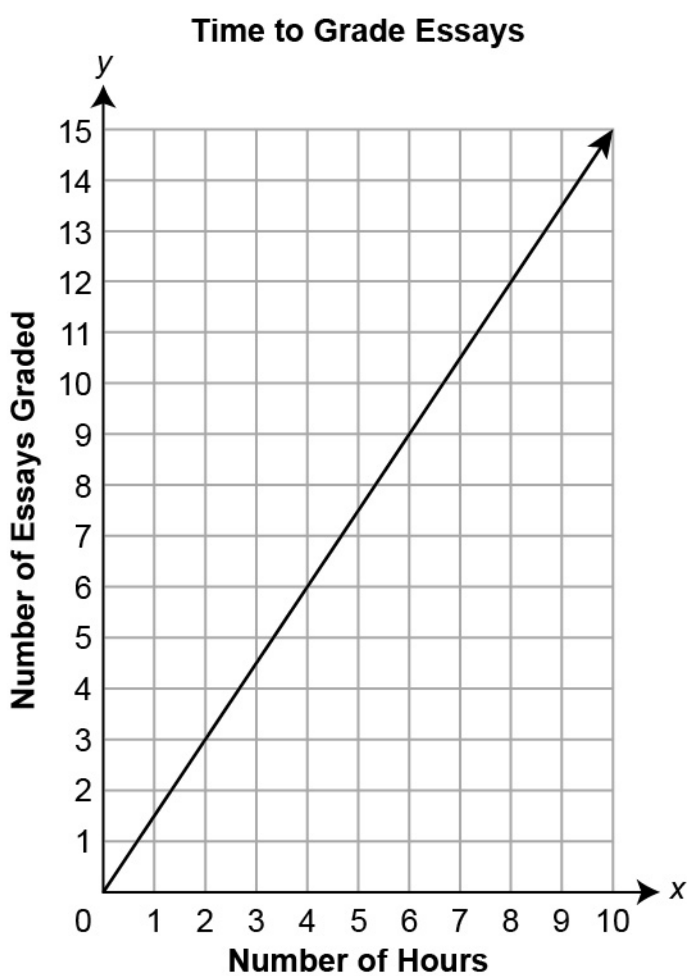
- Ask students to give examples of situations that do not show proportional relationships.

  A restaurant puts 2 rectangular tables of the same size together to fit 10 chairs. If the restaurant put together 3 rectangular tables of the same size, 14 chairs would fit. The relationship between the number of tables and the number of chairs is not proportional because the ratio of tables to chairs in the first case, 2:10, is not equivalent to the ratio of tables to chairs in the second case, 3:14.

How can tables and coordinate planes be used to determine whether a situation is proportional?

M.P.4. Model with mathematics. Identify the defining characteristics of proportional relationships in tables and coordinate planes. For example, identify that the graph of every proportional relationship is a line that contains the origin. Additionally, the ratios of the x- and y-coordinates of points on the graph can be calculated to confirm the proportional relationship.

- Ask students to determine whether a relationship shown in a graph is proportional. For example, the time it takes for Mr. Previs to grade essays is shown in the graph.



Because the relationship creates a graph that is a straight line through the origin, the relationship between number of hours and number of essays graded is a proportional relationship.

- Ask students to determine whether a relationship shown in a table is proportional. For example, give students the following table.

Time (minutes)	20	36	40	42
Pages Read	5	9	10	10.5

The ratios between the time and the number of pages are $\frac{20}{5}$, $\frac{36}{9}$, $\frac{40}{10}$, and $\frac{42}{10.5}$, all of which are equivalent to 4. Therefore, the table shows a proportional relationship.

Key Academic Terms:

proportional, equivalent, ratio, table, coordinate plane, graph, origin, quantity, unit rate, ordered pair, point, relationship

Additional Resources:

- Video: Scale City | Proportional relationships in the real world
- Activity: Scale City | Proportional relationships among time, distance, and speed
- Activity: Thinkport | Proportional relationships and slope: part 1
- Activity: Gym membership plans
- Activity: Ratios and proportional relationships

2b

Proportional Reasoning
Analyze proportional relationships and use them to solve real-world and mathematical problems.
2. Represent a relationship between two quantities and determine whether the two quantities are related proportionally. b. Identify the constant of proportionality (unit rate) and express the proportional relationship using multiple representations including tables, graphs, equations, diagrams, and verbal descriptions.

Guiding Questions with Connections to Mathematical Practices:

Where can the constant of proportionality be found in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships?

M.P.1. Make sense of problems and persevere in solving them. Identify the constant of proportionality, or unit rate, and connect it to different representations of proportional relationships. For example, given the ratio 3:4, identify the constant of proportionality, $\frac{4}{3}$, to create a table with quantities showing the same proportional relationship. Additionally, observe that because there are two quantities that can have a value of 1, there are two values that can be used as a unit rate, each with a different meaning.

- Ask students to explain why the constant of proportionality can be identified by observing which number is paired with a value of 1 in a proportional relationship. For example, in 1 minute Ibram runs 200 yards, in 2 minutes he runs 400 yards, and in 5 minutes he runs 1,000 yards. In a proportional relationship, each pair of values has the same ratio. The constant of proportionality is calculated by finding this common ratio, in this case $\frac{200}{1} = \frac{400}{2} = \frac{1{,}000}{5} = 200$. Using a number (in this case, 200) that is paired with 1 is the constant of proportionality.

- Ask students to identify the constant of proportionality of a proportional relationship that is expressed as a table, graph, equation, or verbal description by identifying the value that is paired with 1. For example, the number of markers that come in x boxes is given by the equation $y = 12x$. To find the number that is paired with 1, substitute $x = 1$ into the equation to get $y = 12$. Because 12 is paired with 1, the constant of proportionality is 12. As a further example, Cierra reads books as shown in the table.

Number of months	0.5	1	1.5	2	2.5
Number of books	1	2	3	4	5

  The number of books that is paired with 1 month is 2. Cierra reads at a unit rate of 2 books per month. Alternately, the number of months that is paired with 1 book is 0.5 months. Cierra's unit rate can also be interpreted as a rate of 0.5 months per book.

- Ask students to create a table or graph that represents a given constant of proportionality. For example, create a table using the ratio $\frac{3}{2}$.

Input	0	1	2	3
Output	0	$\frac{3}{2}$	3	$\frac{9}{2}$

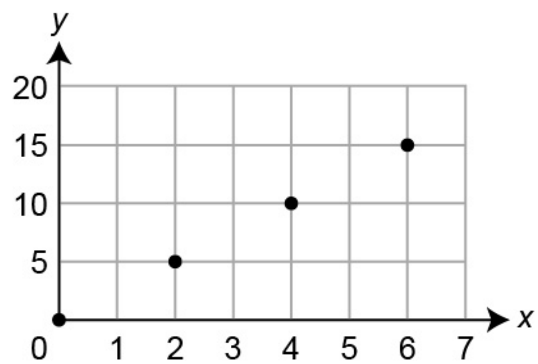
  A proportion must always contain (0, 0), and, because the constant of proportionality in this table is $\frac{3}{2}$, that is the value that is paired with 1. After that, other output values can be found by repeatedly adding $\frac{3}{2}$ to the previous value.

M.P.7. Look for and make use of structure. Determine the constant of proportionality by identifying the ratio between quantities given a graph, equation, diagram, and/or verbal description. For example, in a table where the input values are 0, 1, 2, 3, and 4, and the corresponding output values are 0, 5, 10, 15, and 20, the constant of proportionality is 5. Additionally, use slope triangles rather than the values in the proportion to calculate the constant of proportionality from a graph.

- Ask students to determine the constant of proportionality of a proportional relationship that is represented as a graph, equation, diagram, or description by using ratios. For example, the table shows a relationship between the variables x and y.

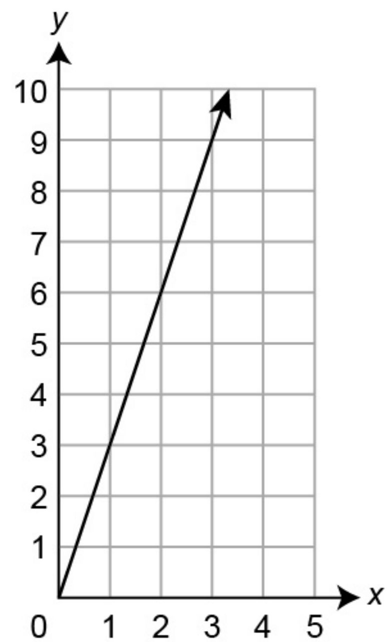
x	3	6	9	12
y	6	12	18	24

The constant of proportionality is 2 because the ratios between each y-value and the corresponding x-value are all equal to 2. As a further example, a proportional relationship is shown in a graph.

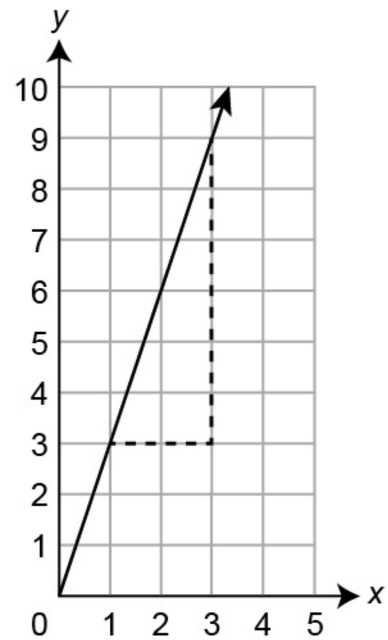


The constant of proportionality can be identified by calculating the ratio between the y-coordinate and x-coordinate of any point. For example, using the rightmost point, the ratio is calculated to be $\frac{15}{6} = \frac{5}{2} = \frac{2.5}{1}$.

- Ask students to determine the constant of proportionality from a graph by creating slope triangles. For example, a proportional relationship is shown.



Ask students to draw a slope triangle by choosing any two points on the graph and drawing a vertical line through one point and a horizontal line through the other point in order to create a right triangle. One possible slope triangle is shown.



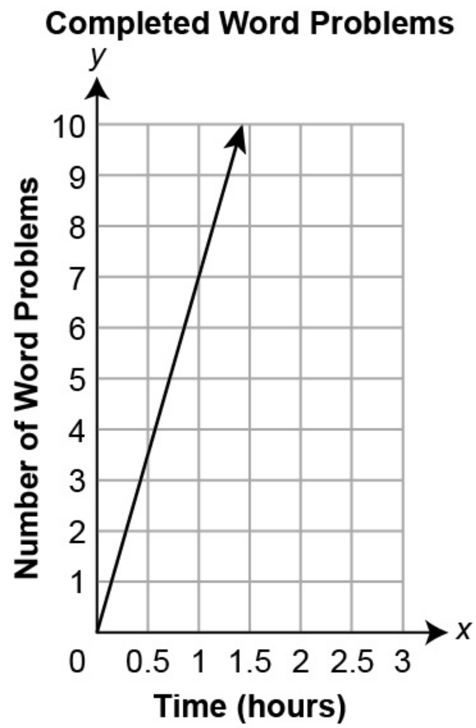
Calculate the constant of proportionality by finding the ratio of the length of the vertical leg of the triangle to the length of the horizontal leg of the triangle. The vertical leg has a length of 6 units and the horizontal leg has a length of 2 units. The ratio, $\frac{6}{2} = \frac{3}{1}$, is the constant of proportionality.

How can an equation be used to represent a proportional relationship?

M.P.8. Look for and express regularity in repeated reasoning. Represent any proportional relationship with an equation in the form $y = kx$, where $k = \frac{B}{A}$ and the ratio B:A is always positive. For example, the problem "Jennie earns $15.00 per hour at her job. How much money, m, does she make after x hours?" can be solved by writing the equation $m = \frac{15}{1}x$ and connecting the equation to a multiplicative comparison. When the unit rate is $15 for exactly one hour, then the total amount of money for x hours is x times as great as 15. Additionally, observe that the constant, k, has the same value as the constant of proportionality.

- Ask students to write an equation to represent a proportional relationship and define the variables involved. For example, an author writes 9 pages of a new book each day. How many total pages, p, does the author write in d days? The equation is $p = 9d$. Because 9 pages are written for every 1 day, the ratio between the quantities is $\frac{9}{1}$, or 9.

- Ask students to write an equation for a proportional relationship represented as a graph. For example, the following graph shows the average number of word problems a student completes in different amounts of time. What equation represents the number of word problems the student completed per hour?



The equation is $c = 7h$, where c is the total number of completed word problems and h is the number of hours the student has been working. The graph contains the point (1, 7), which means that the constant of proportionality is $\frac{7}{1}$, or 7.

Key Academic Terms:

unit rate, constant of proportionality, slope triangle, coordinate plane, multiplicative comparison, proportional relationship, equation, ratio, constant rate of change, ordered pair

Additional Resources:

- Activity: Thinkport | Proportional relationships and slope: part 1
- Activity: Scale City | Similar figures and unknown heights in practice
- Activity: Scale City | Scaling up rectangles using simulations
- Activity: Scale City | Proportional relationships among time, distance, and speed
- Video: Proportional scaling
- Activity: Gym membership plans
- Activity: Proportionality

2c

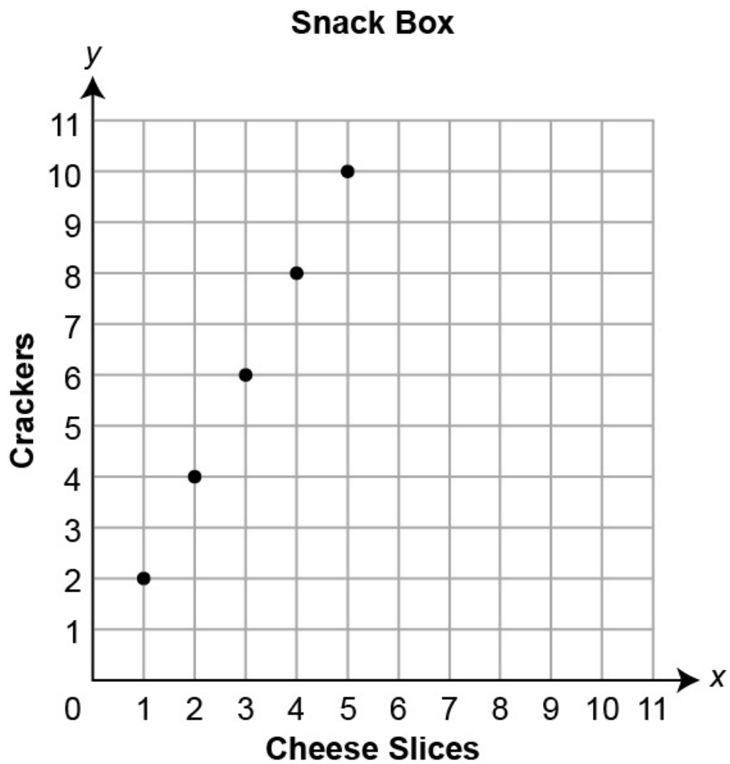
Proportional Reasoning
Analyze proportional relationships and use them to solve real-world and mathematical problems.
2. Represent a relationship between two quantities and determine whether the two quantities are related proportionally. c. Explain in context the meaning of a point (x, y) on the graph of a proportional relationship, with special attention to the points (0, 0) and (1, r) where r is the unit rate.

Guiding Questions with Connections to Mathematical Practices:

How can the context help make sense of a point on a graph of a proportional situation?

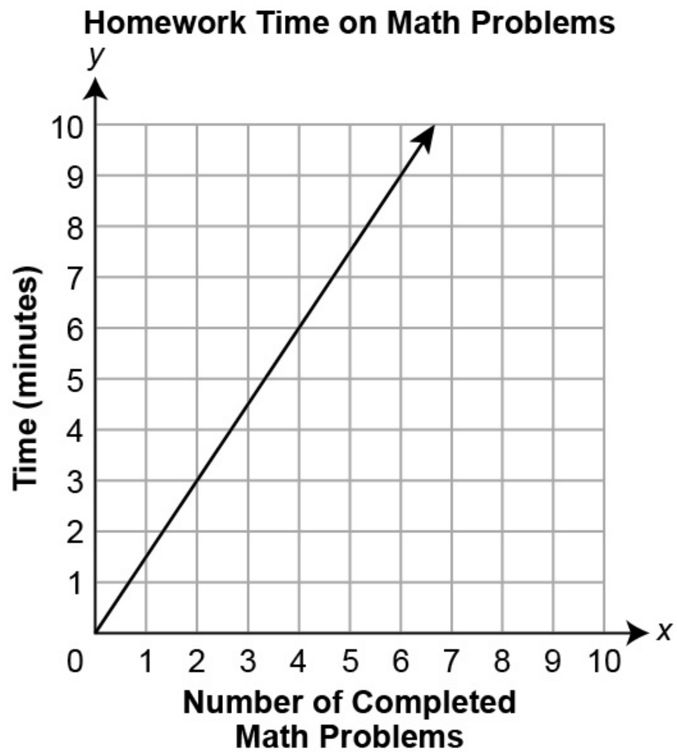
M.P.2. Reason abstractly and quantitatively. Describe the meaning of a point on the graph of a proportional relationship and describe what makes the points (0, 0) and (1, r) important in proportional relationships. For example, given a graph of the situation "for every 3 cups of flour, x, there are 4 tablespoons of butter, y," determine that the point (6, 8) means that if 6 cups of flour are used, then 8 tablespoons of butter are used. And determine that if the ratio of flour to butter is 3:4, then the unit rate is the y-coordinate in the coordinate pair with an x-coordinate of 1, $(1, \frac{4}{3})$, because that point means there are $\frac{4}{3}$ tablespoons of butter for exactly 1 cup of flour. Additionally, identify the unit rate from a given graph and explain the meaning within context.

- Ask students to interpret points on the graph of a proportional relationship. For example, the graph shows the number of cheese slices and crackers that Anita has in her snack box. What does each point on the graph mean?



The x-coordinate of each point represents the number of cheese slices in her snack box and the y-coordinate represents the number of crackers. In particular, the point (2, 4) indicates that when Anita has two slices of cheese, she will also have four crackers. The point (1, 2) not only indicates that when Anita has one slice of cheese, she will also have 2 crackers. Because the point (1, 2) has an x-coordinate value of 1, it also indicates that the proportional relationship has a unit rate of 2 crackers for every 1 cheese slice.

- Ask students to explain how to determine the unit rate from a graph. For example, the following graph shows the amount of time that a student spends on a math problem.



The unit rate in this case indicates the number of minutes that it takes to complete 1 math problem. Because the graph passes through the point $(1, 1\frac{1}{2})$, it takes $1\frac{1}{2}$ minutes to complete 1 math problem. Therefore, the unit rate is $1\frac{1}{2}$ minutes per 1 math problem.

Key Academic Terms:

origin, unit rate, coordinate pair, proportional, constant of proportionality, ordered pair, point, linear, line, *y*-intercept

Additional Resources:

- Lesson: Interpreting points on the graph of a proportional relationship
- Lesson: Grade 7 mathematics module 1, topic B, lesson 10

3

Proportional Reasoning
Analyze proportional relationships and use them to solve real-world and mathematical problems.
3. Solve multi-step percent problems in context using proportional reasoning, including simple interest, tax, gratuities, commissions, fees, markups and markdowns, percent increase, and percent decrease.

Guiding Questions with Connections to Mathematical Practices:

How can proportional relationships be used to solve multistep ratio and percent problems in a variety of ways?

M.P.1. Make sense of problems and persevere in solving them. Solve multistep ratio and percent problems using proportional relationships. For example, solve the word problem "After a 15% discount, the price of a sweater is $42.50. What was the original price of the sweater?" by first finding that 100% – 15% = 85% and then solving the proportion problem "85 to 100 is the same as 42.50 to some number *x*," where *x* is the original price of the sweater. Additionally, ratios and proportions can be used in conjunction to solve multi-step problems.

- Ask students to use proportional relationships to solve a problem involving percentage. For example, give students the problem "Anya drinks 75% of her daily water intake before 4 p.m. She drinks a total of 40 ounces of water a day. How much water, in ounces, w, does Anya drink after 4 p.m.?" Ask students to find the amount of water using proportional relationships. Some sample work is shown.
    - To find the percentage Anya drinks after 4 p.m., subtract 75% from 100% to get 25%.
    - Set up a proportion to show 25% is equal to an unknown quantity of the 40 total ounces of water.

$$\frac{25}{100} = \frac{w}{40}$$

    - Solve the proportion for w to find the number of ounces Anya drinks after 4 p.m. One way to solve the proportion is shown. Other approaches are possible.

$$25 \bullet 40 = 100 \bullet w$$

$$1{,}000 = 100w$$

$$10 = w$$

    - Anya drinks 10 ounces of water after 4 p.m.

How can proportions be used to compare quantities?

M.P.1. Make sense of problems and persevere in solving them. Solve proportion problems, paying attention to the whole, to make comparisons. For example, calculate the percent change in the cost of milk per gallon over a time period and compare it to the percent change in the cost of flour per pound over the same time period to find which product had a greater percent change in price. Additionally, proportions can be used to compare the percent change in a quantity.

- Ask students to solve a proportion to determine a percent increase or percent decrease. For example, give students the problem “Tyler started running for exercise. The first day, he ran 1 mile in 12 minutes. After several months of training, Tyler ran 1 mile in 9 minutes. What is the percent decrease in the number of minutes it takes for Tyler to run 1 mile?” Use the whole of 12 minutes to find the percent decrease. Some sample work is shown.
    - First, find the decrease in Tyler’s time by subtracting 9 from 12. Then, set up a proportion to compare the decrease to the whole and use the variable d to represent the percent decrease.

$$\frac{3}{12} = \frac{d}{100}$$

    - Next, solve the proportion to find the percent decrease in Tyler’s time to run one mile. One way to solve the proportion is shown. Other approaches are possible.

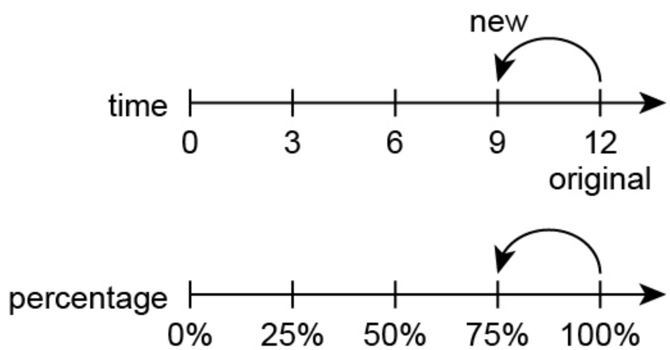
$$3 \bullet 100 = 12 \bullet d$$

$$300 = 12d$$

$$25 = d$$

    - Tyler decreased his time to run one mile by 25%.

Similarly, a double number line can be used to identify the percent decrease. For the example above, two number lines can be created showing proportional relationships: one between the time, in minutes, it originally took Tyler to run 1 mile and the time, in minutes, it took him after several months of training and the other between the percentage representing his original amount of time (100%) and the percentage representing his new amount of time in relation to his original amount of time.



Since the difference between the time it took Tyler to run 1 mile originally and the time it took him to run 1 mile after several months of training is 3 minutes, which is $\frac{1}{4}$ of the original time of 12 minutes, then the percent decrease is equal to $\frac{1}{4}$ of 100%, or a 25% decrease.

- Ask students to use percent increase or decrease to compare two quantities. For example, give students the situation "A clothing store is having a sale. The price of pants is reduced from $30.00 to $25.50. The price of shirts is reduced from $12.50 to $10.00. Which clothing item has a greater percent discount?" Ask students to find the percent discount for each item to determine which has the greater discount. Some sample work is shown.
  - Before finding the percent discount for each item, first find the price discount by subtracting the reduced price from the original price.

    For pants, $30 - 25.5 = 4.5$, so the pants are reduced by $4.50.

    For shirts, $12.5 - 10 = 2.5$, so the pants are reduced by $2.50.
  - Next, determine the percent discount by using proportions.

    The proportion for pants is $\frac{4.5}{30} = \frac{p}{100}$ where p is the percent discount for pants.

    The proportion for shirts is $\frac{2.5}{12.5} = \frac{s}{100}$ where s is the percent discount for shirts.
  - Solving the proportions for p and s show that the discount for pants is 15% and the discount for shirts is 20%. Therefore, shirts have the greater percent discount.

How is the whole determined in percent problems?

M.P.2. Reason abstractly and quantitatively. Analyze a problem to determine the whole to solve percent problems. For example, when solving the problem "Jude makes 8 quarts of fruit punch. The fruit punch is made by mixing equal parts of orange juice and lemonade. The lemonade is $\frac{1}{5}$ lemon juice and $\frac{4}{5}$ sugar water. What percentage of the fruit punch is sugar water?" note that the lemonade is half of the whole that is 8 quarts, that the sugar water is $\frac{4}{5}$ of the 4 quarts of lemonade, and that the solution wants the percentage to be out of the whole of 8 quarts. Additionally, know that some problems may have multiple wholes.

- Ask students to use proportions to find the whole in a percent problem. For example, give students the problem "Elly pays \$21.20 for a toy at the store. The sales tax is 6% and is included in the total Elly pays. What was the price of the toy before sales tax?" Determine that the whole in this problem is the price of the toy before sales tax. A proportion can be used to determine this price, m. Some sample work is shown.
    - First, use 21.20 – m to represent the amount of sales tax in terms of m, where m is the price of the toy before sales tax, and use that as part of the proportion.

$$\frac{21.20 - m}{m} = \frac{6}{100}$$

    - Then, solve for m to find the price of the toy before sales tax.

$$(21.20 - m) \bullet 100 = m \bullet 6$$

$$2{,}120 - 100m = 6m$$

$$2{,}120 = 106m$$

$$20 = m$$

    - The price of the toy before sales tax was \$20.00.

Proportional reasoning could also be used to determine the price of the toy before sales tax. Since \$21.20 is equal to a 6% increase to the price of the toy before sales tax, a proportion can be set up using ratios comparing the price with sales tax to the price before sales tax in both dollars and percentages.

$$\frac{21.20}{m} = \frac{106}{100}$$

Solving this proportion for m results in $m = 20$, which means the price of the toy before sales tax was \$20.

- Ask students to solve problems involving two different wholes. For example, give students the problem "Charlie makes 6 quarts of a cleaning solution. The solution is $\frac{3}{4}$ water and $\frac{1}{4}$ soap mixture. The soap mixture is $\frac{1}{2}$ liquid soap and $\frac{1}{2}$ vinegar. How many quarts of liquid soap does Charlie use to make the cleaning solution?" The first whole in the problem is the 6 quarts of cleaning solution, and the other whole is the soap mixture. Some sample work is shown.
  - The soap mixture is one-fourth of the 6 quarts, and one-fourth is the same as 25%, so the soap mixture is 25% of 6 quarts, or $0.25 \bullet 6 = 1.5$ quarts.
  - The liquid soap is one-half of the soap mixture, and one-half is the same as 50%, so the liquid soap is 50% of 1.5 quarts, or $0.5 \bullet 1.5 = 0.75$ quarts.
  - Charlie uses 0.75 quarts of liquid soap to make the cleaning solution.

Key Academic Terms:

percent, interest, tax, markup, markdown, gratuity, commission, fee, percent increase, percent decrease, percent error, proportional, ratio

Additional Resources:

- Activity: Toy design
- Video: Proportional relationships
- Lesson: Use proportions to solve percent problems
- Video: Proportions and percent

4a

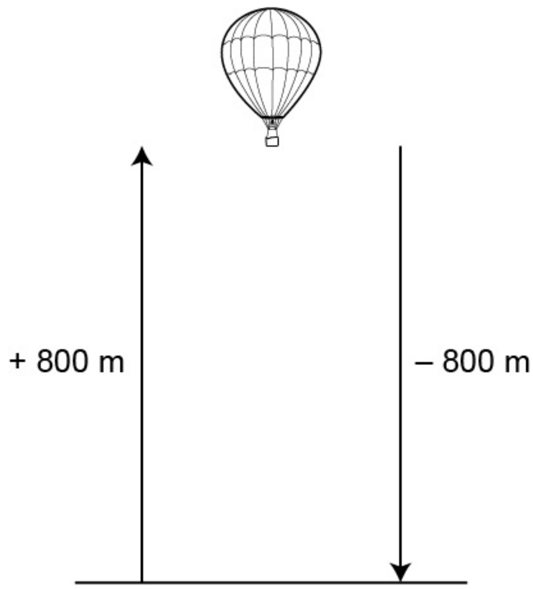
Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
4. Apply and extend knowledge of operations of whole numbers, fractions, and decimals to add, subtract, multiply, and divide rational numbers including integers, signed fractions, and decimals. a. Identify and explain situations where the sum of opposite quantities is 0 and opposite quantities are defined as additive inverses.

Guiding Questions with Connections to Mathematical Practices:

When do two quantities combine to make zero?

M.P.2. Reason abstractly and quantitatively. Observe when a sequence of events results in no change in a given context. For example, if an elevator starts on the ground floor, goes up to the 8th floor (+8), and then returns to the ground floor (–8), the end result is the same as the beginning of the situation because $8 - 8 = 0$. Additionally, observe that if two numbers in a situation combine to make 0, then the two numbers are additive inverses.

- Ask students to write an equation to represent a context involving additive inverses. For example, tell students about a hot air balloon that ascends from the beach to an altitude of 800 meters before descending back to the beach. Students describe the situation with the equation $800 + (-800) = 0$.

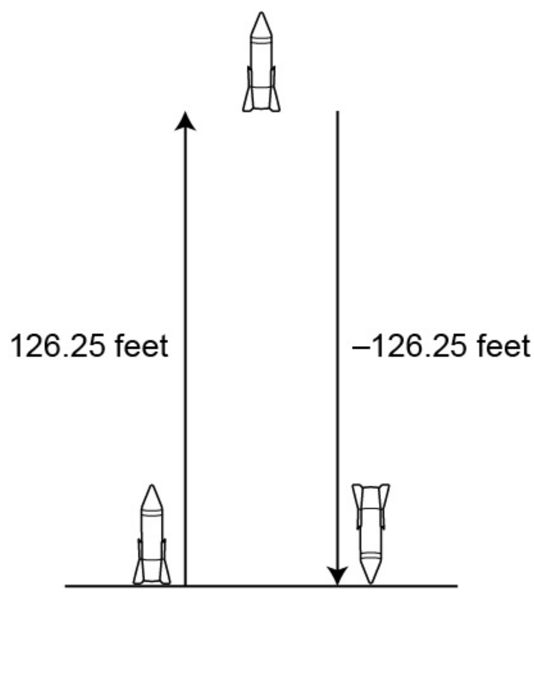


- Provide students with a rational number and ask them to identify its additive inverse. Then, ask students to create a context that could represent the number and its inverse. For example, if given the number 22.5, students should be able to identify the additive inverse as –22.5. A possible context for these numbers could be depositing $22.50 into a bank account and then withdrawing $22.50 several days later to make a purchase.

How can the sum of a number and its additive inverse be used to describe real-world contexts?

M.P.4. Model with mathematics. Represent a real-world situation with rational numbers to show a result of 0. For example, the location of a submarine that is 700 feet below sea level can be represented by the number –700. If the submarine rises 700 feet, then its new location can be represented by adding 700 to –700, or $-700 + 700 = 0$. The sum of 0 represents sea level in this context. Additionally, draw a diagram or picture to illustrate that the sum of a number and its additive inverse is zero.

- Ask students to create a context that corresponds to an equation that shows the sum of a number and its additive inverse. For example, start with the equation $\frac{5}{8} + (-\frac{5}{8}) = 0$.
  A student might imagine a situation in which $\frac{5}{8}$ gallon of gas is poured into the tank of a lawn mower. The lawn mower is then used until the tank is empty. The equation $\frac{5}{8} + (-\frac{5}{8}) = 0$ represents the amount of gas in the tank after the mower is filled and used.
- Ask students to draw a picture and create an equation when provided with a context that involves the sum of a number and its additive inverse. For example, a model rocket is launched from the ground and ascends to an altitude of 126.25 feet. The rocket then descends back to the ground. Students represent this context with the equation $126.25 + (-126.25) = 0$ and a drawing like the one shown.



Key Academic Terms:

rational number, number line, opposite, addition, subtraction, positive, negative, horizontal, vertical, additive inverse

Additional Resources:

- Video: Addition of positive and negative integers
- Video: Competing for the Northern Hemisphere Games
- Video: Thinkport | Opposite quantities and zero pairs
- Activity: Bookstore account
- Activity: Distances on the number line 2

4b

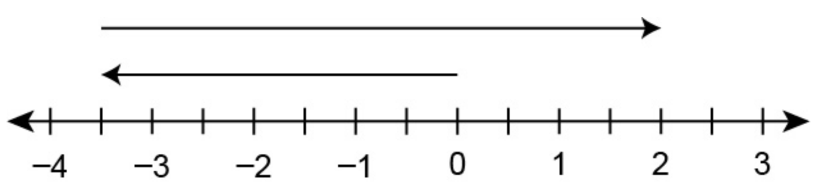
Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
4. Apply and extend knowledge of operations of whole numbers, fractions, and decimals to add, subtract, multiply, and divide rational numbers including integers, signed fractions, and decimals. b. Interpret the sum of two or more rational numbers, by using a number line and in real-world contexts.

Guiding Questions with Connections to Mathematical Practices:

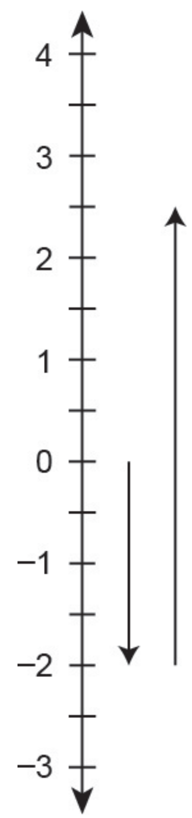
How can a number line model be used to add rational numbers?

M.P.4. Model with mathematics. Represent rational numbers on a horizontal number line with positive numbers to the right of 0 and negative numbers to the left of 0. Use directional arrows to combine numbers. For example, $5 + (-4)$ can be represented on a number line with an arrow starting at 0 and pointing 5 units to the right and then another arrow starting at 5 and pointing 4 units to the left from 5 to 1. This shows that $5 + (-4) = 1$. Additionally, use a vertical number line with positive numbers above zero and negative numbers below zero to represent the sum of rational numbers.

- Ask students to create an equation that corresponds to a given number line with directional arrows. For example, given the number line shown, generate the equation $-3.5 + 5.5 = 2$.



- Ask students to use a vertical number line to determine a sum given a context that involves rational numbers. For example, suppose the temperature in a freezer at 6:00 a.m. was –2°C. The temperature then increased 4.5°C by 4:00 p.m. What was the temperature in the freezer at 4:00 p.m.?



Students should determine that the temperature at 4:00 p.m. was 2.5°C.

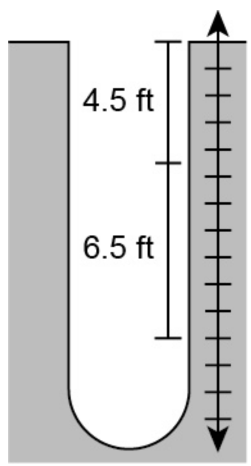
- Ask students to find the distance on the number line between a sum of two numbers and the first addend of the sum. For example, find the distance between $3 + 7$ and 3 and find the distance between $3 + (-7)$ and 3. Observe that in either case, both $3 + 7$ and $3 + (-7)$ are 7 units from 3. The point $3 + 7$ is a distance of $|7|$ units from 3 in the positive direction and the point $3 + (-7)$ is a distance of $|-7|$ units from 3 in the negative direction.

How can sums of rational numbers be used to describe real-world contexts?

M.P.4. Model with mathematics. Assign a rational number to an appropriate situation where there is an increase or decrease. For example, if a thermostat was set at 75 degrees and is then turned down 7 degrees, the situation can be represented by the expression 75 – 7 or 75 + (–7). Additionally, draw pictures or diagrams that represent the sum or difference of rational numbers in real-world contexts.

- Provide students with an expression and ask them to create a corresponding real-world context. For example, given the expression $9\frac{1}{2} + (-3)$, a student might imagine a situation where someone climbs $9\frac{1}{2}$ feet up a ladder to clean the top of a window then descends 3 feet down the ladder to clean the bottom of the window.

- Provide students with a real-world context and ask them to model it with a picture and expression. For example, Tucker lowered a bucket 4.5 feet down a well, paused for a moment, and then lowered the bucket an additional 6.5 feet. Students could represent this context with the picture shown and the expressions –4.5 – 6.5 or –4.5 + (–6.5).



Key Academic Terms:

rational, number line, opposite, addition, subtraction, positive, negative, absolute value, additive inverses, context

Additional Resources:

- Activity: The number system
- Lesson: Football, absolute value and a life-size number line too
- Video: Addition of positive and negative integers
- Video: Competing for the Northern Hemisphere Games
- Lesson: A concrete introduction to the abstract concepts of integers and algebra using algebra tiles

4c

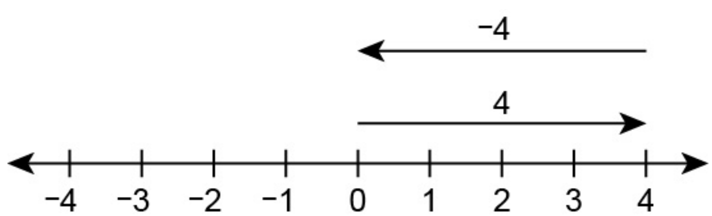
Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
4. Apply and extend knowledge of operations of whole numbers, fractions, and decimals to add, subtract, multiply, and divide rational numbers including integers, signed fractions, and decimals. c. Explain subtraction of rational numbers as addition of additive inverses.

Guiding Questions with Connections to Mathematical Practices:

How can the additive inverse of a rational number help to solve an addition problem?

M.P.7. Look for and make use of structure. Observe that a number and its additive inverse combine to make zero. For example, if an arrow on a number line starts at 0 and moves to the right $\frac{3}{5}$ of a unit, and a second arrow starts at $\frac{3}{5}$ and moves to the left $\frac{3}{5}$ of a unit, then the two arrows overlap exactly and show that $\frac{3}{5} + (-\frac{3}{5}) = 0$. Additionally, use the additive inverse to determine a missing addend when the sum of two numbers is equal to zero.

- Ask students to demonstrate why a number and its additive inverse add to 0. For example, $4 + (-4)$ can be modeled on a number line by starting at 0 and moving 4 units in the positive direction (right), followed by moving 4 units in the negative direction (left).



  The result is ending where the model began at 0. Therefore, $4 + (-4) = 0$.

- Provide students with a number and an unknown number whose sum is equal to zero. Then, ask them to determine the unknown number and justify the answer. For example, given the equation $0.47 + \square = 0$, students should be able to determine that the missing number is -0.47 because $0.47 + (-0.47) = 0$.

- Provide students with a set of three numbers—zero and two additive inverses. Then, ask students to arrange the numbers so that the sum of two numbers is equal to the remaining number. For example, given the numbers $-\frac{2}{7}$, 0, and $\frac{2}{7}$, students should create an equation in the form ___ + ___ = ___. In this case, $\frac{2}{7} + -\frac{2}{7} = 0$ or $-\frac{2}{7} + \frac{2}{7} = 0$.

How can a subtraction problem be changed into an addition problem?

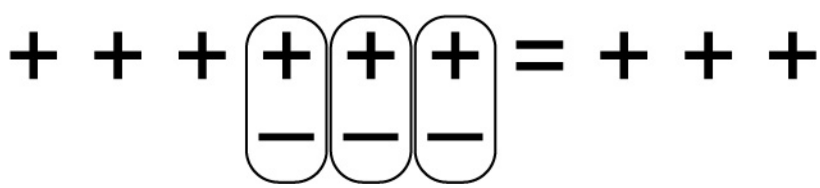
M.P.1. Make sense of problems and persevere in solving them. Represent any subtraction problem as adding the additive inverse by connecting the "take away" model of subtraction to the idea of adding a negative number. For example, "10 take away 8" can be modeled as 10 – 8 = 2, and 10 + –8 can be modeled on a number line as adding (–8) to 10 with an arrow starting at 10 that points 8 units left and ends at 2. Therefore 10 – 8 = 10 + (– 8). Additionally, extend the concept of subtraction as adding the additive inverse to include subtraction of a negative number.

- Ask students to rewrite a subtraction problem involving a negative number by making it an addition problem. For example, given the expressions 4 – (–2), –10 – (–16), and –3.75 – (–0.75), student responses may include 4 + 2, –10 + 16, and –3.75 + 0.75.

- Ask students to model a subtraction problem with a "take away" drawing. For example, 6 – 3 can be modeled as 6 "positives" with 3 taken away or crossed off.

  $$+ \ + \ + \ \cancel{+} \ \cancel{+} \ \cancel{+} = + \ + \ +$$

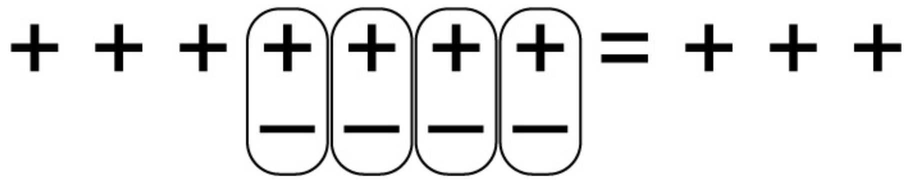
  Then, ask students to model the same problem with a drawing that shows adding the additive inverse.

  The models show that both 6 – 3 and 6 + (–3) are equal to 3.

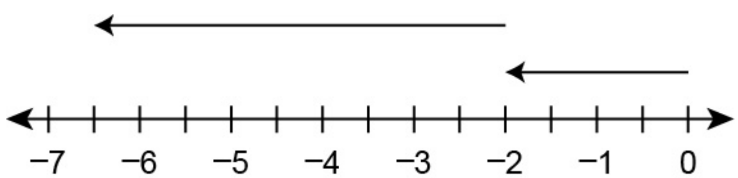
M.P.1. Make sense of problems and persevere in solving them. Interpret any addition problem as subtracting the additive inverse of the second addend. For example, –215 + (–62) can be rewritten as –215 – 62. Additionally, represent subtraction of an additive inverse using a number line with directional arrows.

- Ask students to write a subtraction statement given a model that represents addition. For example, provide students with the model shown.

  Observe that the model showing 7 + (–4) is equivalent to 7 – 4.

- Provide students with an addition problem that can be rewritten by subtracting the additive inverse. Ask students to rewrite and model the problem with a number line and directional arrows. For example, given the problem –2 + (–4.5), students rewrite it as –2 – 4.5 and represent it on a number line with directional arrows as shown.

How can the properties of operations be used to add and subtract rational numbers?

M.P.2. Reason abstractly and quantitatively. Apply the properties of operations to problems involving rational numbers. For example, find the value of the expression $-17\frac{1}{2} + 84 - (-\frac{3}{4})$ by rewriting it as $84 + \frac{3}{4} - 17\frac{1}{2}$ and then evaluating from left to right. Additionally, use properties of operations to add and subtract rational numbers to solve real-world problems.

- Ask students to write an expression that includes addition and subtraction of rational numbers at least two different ways. For example, given the expression $6\frac{1}{3} - 4\frac{2}{3} - 2\frac{1}{3}$, students should rewrite the expression as either $6\frac{1}{3} - (4\frac{2}{3} + 2\frac{1}{3})$ or as $(6\frac{1}{3} - 4\frac{2}{3}) + (-2\frac{1}{3})$. The value of the expression is $-\frac{2}{3}$.
- Ask students to write an expression that corresponds to a contextual problem involving the addition and subtraction of rational numbers and then evaluate. For example, Sarah received $20 for her birthday. She used the money to purchase socks for $3.50 and a headband for $2.75. How much money does Sarah have after her purchases? The expressions $20 - 3.5 - 2.75$ and $20 - (3.5 + 2.75)$ can be used to determine that Sarah has $13.75 after her purchases.
- Ask students to find the value of an expression by creating pairs of additive inverses. For example, ask students to calculate $7 + (-3)$. Students should represent 7 as $(4 + 3)$, which makes the equation $(4 + 3) + (-3)$. By the associative property, students can add the 3 and -3 first, making the expression $4 + 0$, which has a value of 4. Further, this concept can be used to algebraically demonstrate why subtracting a number has the same effect as adding the additive inverse.

Key Academic Terms:

rational, number line, opposite, addition, subtraction, positive, negative, absolute value, additive inverses, context, distance, properties of operations

Additional Resources:

- Activity: The number system
- Lesson: Football, absolute value and a life-size number line too
- Video: Addition of positive and negative integers
- Video: Competing for the Northern Hemisphere Games
- Lesson: A concrete introduction to the abstract concepts of integers and algebra using algebra tiles
- Lesson: Subtracting integers—How does subtraction relate to addition?
- Video: Food miles: absolute values to the rescue
- Activity: Differences and distances
- Activity: Differences of integers
- Video: Subtracting rational numbers
- Activity: Distances between houses
- Lesson: Discovering how to subtract rational numbers using the additive inverse
- Video: Subtracting positive and negative numbers
- Video: Using lure of the labyrinth in your classroom
- Lesson: Applying the properties of operations to add and subtract rational numbers

4d

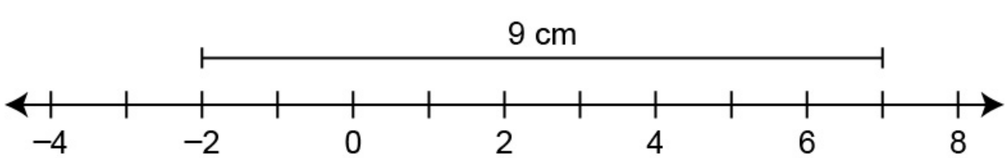
Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
4. Apply and extend knowledge of operations of whole numbers, fractions, and decimals to add, subtract, multiply, and divide rational numbers including integers, signed fractions, and decimals. d. Use a number line to demonstrate that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

Guiding Questions with Connections to Mathematical Practices:

How can absolute value be used to model distance on a number line?

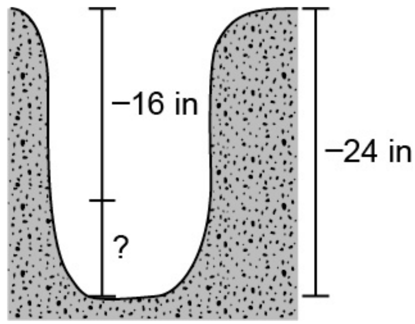
M.P.4. Model with mathematics. Observe that the difference model for subtraction can be combined with absolute value to represent distance. The distance is the length of the interval between points. Unlike location on a number line, distance is not directional; therefore, distance is never negative. Absolute value is useful for modeling distance because it is also never negative. For example, to find the distance between −9 and 5.5 on a number line, find the difference between −9 and 5.5 by subtracting and then find the absolute value of the difference. Note that the absolute value of 5.5 – (−9) gives the same distance. Additionally, explain the connection between absolute value when subtracting and distance in real-world contexts.

- Provide students with a ruler and a number line that has tick marks spaced 1 centimeter apart. Then, ask students to find the distance between two numbers by measuring with the ruler. Next, ask the students to evaluate an expression representing the distance between the numbers using absolute value. For example, given the numbers −2 and 7, students measure the distance between the numbers as 9 centimeters.



Likewise, −2 – 7 = −9, and |−9| = 9.

- Provide students with a context that involves distance and can be represented with subtraction. Then, ask students to draw a picture to illustrate the meaning of the subtraction and the significance of absolute value. For example, a fence installer dug a hole 16 inches deep, took a break, then continued digging. The depth of the hole, when finished, was 24 inches. How much deeper did the fence installer dig after taking a break? Represent this situation with a drawing and the expression $|-16 - (-24)| = 8$, noting that the distance between the two depths is 8 inches.



Key Academic Terms:

rational, number line, opposite, addition, subtraction, positive, negative, absolute value, additive inverses, context, distance

Additional Resources:

- Activity: The number system
- Lesson: Football, absolute value and a life-size number line too
- Video: Addition of positive and negative integers
- Video: Competing for the Northern Hemisphere Games
- Lesson: A concrete introduction to the abstract concepts of integers and algebra using algebra tiles
- Lesson: Subtracting integers—How does subtraction relate to addition?
- Video: Food miles: absolute values to the rescue
- Activity: Differences and distances
- Activity: Differences of integers
- Video: Subtracting rational numbers
- Activity: Distances between houses
- Lesson: Discovering how to subtract rational numbers using the additive inverse

4e

Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
4. Apply and extend knowledge of operations of whole numbers, fractions, and decimals to add, subtract, multiply, and divide rational numbers including integers, signed fractions, and decimals. e. Extend strategies of multiplication to rational numbers to develop rules for multiplying signed numbers, showing that the properties of the operations are preserved.

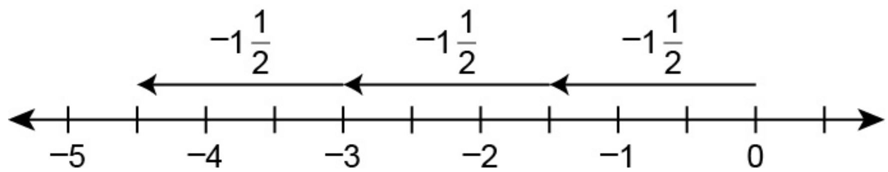
Guiding Questions with Connections to Mathematical Practices:

How can the meaning of multiplication, strategies for multiplying fractions, and the properties of operations be used to solve multiplication problems involving rational numbers?

M.P.8. Look for and express regularity in repeated reasoning. Apply prior knowledge of multiplication and multiplication strategies to solve rational number multiplication problems. For example, $\frac{1}{3} \times (-6)$ can be decomposed into $\frac{1}{3} \times 6 \times (-1)$ or $(-1) \times (\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3})$. Additionally, use number lines to represent and determine products of rational numbers.

- Provide students with a multiplication problem involving two positive numbers and ask them to write an equivalent problem using repeated addition. Then, change the sign of one number in the problem and ask students to write the problem with repeated addition, noting the sign difference between the two problems. For example, $5 \times \frac{1}{4}$ can be written as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, which is equivalent to $\frac{5}{4}$. The problem $5 \times (-\frac{1}{4})$ can be written as $(-\frac{1}{4}) + (-\frac{1}{4}) + (-\frac{1}{4}) + (-\frac{1}{4}) + (-\frac{1}{4})$, which is equivalent to $-\frac{5}{4}$.

- Provide students with a multiplication problem involving an integer and a rational number. Then, ask students to represent the problem by drawing a number line with directional arrows. Use the result to determine the product. For example, given the problem $3 \times (-1\frac{1}{2})$, a student draws three consecutive arrows that are each $-1\frac{1}{2}$ units long to determine a product of $-4\frac{1}{2}$.



How can the sign of a product be determined?

M.P.8. Look for and express regularity in repeated reasoning. Use models and the properties of operations to develop rules for multiplying signed rational numbers. For example, observe the patterns of 5: $5 \times 2 = 10$, $5 \times 1 = 5$, $5 \times 0 = 0$, so the next integer to multiply by 5 should give a product of 5 less than the previous product to follow the pattern; therefore, $5 \times (-1) = -5$, $5 \times (-2) = -10$, and so on. Additionally, extend patterns of products vertically and horizontally within a table to visualize that the product of two negative numbers is positive.

- Provide students with a multiplication equation that has a missing sign and ask students to determine and justify what the sign should be. For example, given the equation shown, conclude that the sign of $\frac{1}{6}$ must be negative because a product is negative when the signs of the factors are different.

$$\boxed{+}\,\frac{1}{5} \times \boxed{?}\,\frac{1}{6} = -\frac{1}{30}$$

- Provide students with a multiplication table and ask them to fill out the products of two positive numbers or any number and 0.

×	**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**
3				0	3	6	9
2				0	2	4	6
1				0	1	2	3
0	0	0	0	0	0	0	0
−1				0			
−2				0			
−3				0			

Then, ask students to extend the patterns so that the products of a positive number and a negative number are revealed.

×	**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**
3	−9	−6	−3	0	3	6	9
2	−6	−4	−2	0	2	4	6
1	−3	−2	−1	0	1	2	3
0	0	0	0	0	0	0	0
−1				0	−1	−2	−3
−2				0	−2	−4	−6
−3				0	−3	−6	−9

Finally, extend the patterns to completely fill out the table.

×	**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**
3	−9	−6	−3	0	3	6	9
2	−6	−4	−2	0	2	4	6
1	−3	−2	−1	0	1	2	3
0	0	0	0	0	0	0	0
−1	3	2	1	0	−1	−2	−3
−2	6	4	2	0	−2	−4	−6
−3	9	6	3	0	−3	−6	−9

Note that when the signs of the factors are the same, the product is positive; when the signs of the factors are different, the product is negative.

M.P.7. Look for and make use of structure. Apply the distributive property to demonstrate why multiplying two negative integers gives a positive product. For example, decomposing 4 and -4 as $1 \times (5 + (-1)) = 4$ and $(-1) \times (5 + (-1)) = -4$ and then applying the distributive property means that $(-1) \times 5 + (-1)(-1) = -4$ and $-5 + (-1)(-1) = -4$; therefore, $(-1)(-1)$ must equal 1. Additionally, use knowledge of additive inverses to show that $(-1) \times (-1) = 1$.

- Ask students to assist in demonstrating that $(-1) \times (-1) = 1$ by providing justification for mathematical steps. For example, demonstrate the justification of the following statements.

Statement	**Justification**
$1 + (-1) = 0$	additive inverses
$-1 \times 0 = 0$	zero product property
$-1 \times (1 + (-1)) = 0$	substitution
$(-1)(1) + (-1)(-1) = 0$	distributive property
$-1 + (-1)(-1) = 0$	identity property
$(-1)(-1) = 1$	additive inverses

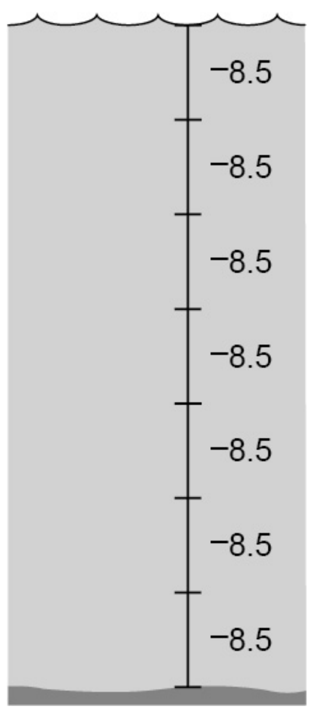
- After demonstrating that the product of (-1) and (-1) is 1, ask students to use that fact to assist in showing why the product of any two negative numbers is positive. For example, begin with the product $(-5)(-7)$. Ask students to write both (-5) and (-7) as a product of two numbers. In this case, (-5) can be written as $(-1)(5)$ and (-7) can be written as $(-1)(7)$. As such, $(-5)(-7)$ is equivalent to $(-1)(5)(-1)(7)$. Using the commutative property and the associative property, students can write $(-1)(5)(-1)(7)$ as $(-1)(-1)(5)(7)$. Since the product of (-1) and (-1) is 1, the expression is also equivalent to $(1)(5)(7)$, which has a positive product of 35.

How can real-world situations be used to interpret products of rational numbers?

M.P.2. Reason abstractly and quantitatively. Use a real-world situation to contextualize a multiplication equation to interpret and make meaning out of negative and positive products. For example, the equation $3 \times (-1.2) = (-3.6)$ could represent a temperature drop of about 1.2 degrees every hour for 3 hours for a total temperature drop of 3.6 degrees. Additionally, draw pictures or diagrams that can be used to represent real-world products involving negative numbers.

- Provide students with a multiplication equation and ask them to create a corresponding real-world situation. For example, given the equation $5 \times (-1.25) = -6.25$, a student might imagine a situation in which a person owes his or her friend \$6.25 for 5 baseball cards that are each worth \$1.25.

- Provide students with a real-world situation that can be modeled with a multiplication equation in which either the multiplicand or the multiplier is a negative number. Then, ask students to write the equation and draw a picture or diagram. For example, a baited fishhook dropped in a pond descends 8.5 inches per second for 7 seconds until it reaches the bottom. Students could write the equation $7 \times (-8.5) = -59.5$ and draw a picture like the one shown.



Key Academic Terms:

multiply, divide, rational, fraction, properties of operations, integer, product, positive, negative, compose, decompose

Additional Resources:

- Activity: Why is a negative times a negative always positive?
- Activity: Distributive property of multiplication

4f

Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
4. Apply and extend knowledge of operations of whole numbers, fractions, and decimals to add, subtract, multiply, and divide rational numbers including integers, signed fractions, and decimals. f. Divide integers and explain that division by zero is undefined. Interpret the quotient of integers (with a non-zero divisor) as a rational number.

Guiding Questions with Connections to Mathematical Practices:

How can the meaning of division, the relationship between multiplication and division, strategies for dividing fractions, and the properties of operations be used to solve division problems involving rational numbers?

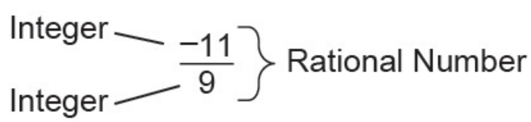
M.P.8. Look for and express regularity in repeated reasoning. Apply prior knowledge of division and division strategies to solve rational number division problems. For example, to solve $5 \div \frac{2}{3}$, make a pictorial representation of 5 wholes and then partition each whole into $\frac{2}{3}$ sections with sections overlapping the wholes. There are 7 sections plus half of another section, so the solution to $5 \div \frac{2}{3}$ is $7\frac{1}{2}$. In addition, to solve a problem like $\frac{-8}{9} \div \frac{4}{3}$, another strategy that works when the numerators and denominators of fractions divide easily is to divide the numerators, $-8 \div 4 = -2$, and then divide the denominators, $9 \div 3 = 3$, for a solution of $\frac{-2}{3}$. Further, use the relationship between multiplication and division to solve a division equation by writing it as an equivalent multiplication equation.

- Provide students with a real-world or mathematical division equation and ask them to solve it by writing it as a multiplication equation. For example, the equation $6 \div \frac{1}{4} = \square$ can be written as $\square \times \frac{1}{4} = 6$ and thought of as asking the question "How many one-fourths are needed to have 6?" Since $\frac{1}{4}$ taken 24 times is 6, it can be concluded that $6 \div \frac{1}{4} = 24$.

- Provide students with a division problem and ask them to solve it using two different methods. For example, the problem $(\frac{12}{15}) \div (\frac{3}{-5})$ can be solved by finding the quotient of the numerators and the quotient of the denominators such that $\frac{12 \div 3}{15 \div -5} = \frac{4}{-3} = -\frac{4}{3}$. Alternatively, solve the problem by multiplying the dividend by the reciprocal of the divisor such that $\frac{12}{15} \times \frac{-5}{3} = \frac{12 \times -5}{15 \times 3} = \frac{-60}{45} = \frac{-4}{3} = -\frac{4}{3}$.

M.P.6. Attend to precision. Know and apply the definitions of “integer” and “rational number.” For example, $\frac{3}{8}$ is a rational number because it is the quotient of the integers 3 and 8 and the denominator is not zero. Additionally, write a quotient expression as a fraction and a fraction as a quotient expression.

- Provide students with a fraction and ask them to apply labels to each part and the whole. For example, given the fraction $\frac{-11}{9}$, apply the label “integer” to −11 and 9 and the label “rational number” to the fraction.



- Provide students with a division problem and ask them to rewrite it as a fraction. For example, given the problem 16 ÷ 18, write either $\frac{16}{18}$ or $\frac{8}{9}$.

- Provide students with a number and ask them to determine whether it is an integer, a rational number, or both. For example, −4 is an integer and a rational number because it can be thought of as −4 ÷ 1 or $-\frac{4}{1}$. By contrast, $-\frac{1}{4}$ is a rational number but not an integer because it is neither a whole number nor the opposite of a whole number.

How can the sign of a quotient be determined?

M.P.8. Look for and express regularity in repeated reasoning. Use models and the properties of operations to develop rules for dividing signed rational numbers. For example, since $-6 \times -3 = 18$, it follows that $18 \div (-6) = -3$. Additionally, know that if the signs of the dividend and divisor are the same, then the sign of the quotient is positive; if the signs of the dividend and divisor are different, then the sign of the quotient is negative.

- Provide students with division problems and ask them to determine the sign of the quotients. For example, given the problem $-\frac{7}{13} \div \frac{15}{11}$, identify that the sign of the quotient will be negative.

- Provide students with the signs of either the dividend and the divisor, the dividend and the quotient, or the divisor and the quotient and ask them to determine the sign of the remaining component. For example, given a negative divisor and a negative quotient, determine that the sign of the dividend must be positive. Given a negative dividend and negative divisor, determine that the sign of the quotient must be positive.

- Provide students with a division equation and ask them to determine three related division or multiplication equations. For example, given the equation $(\frac{1}{3}) \div (-\frac{1}{2}) = (-\frac{2}{3})$, produce the following equations: $(\frac{1}{3}) \div (-\frac{2}{3}) = (-\frac{1}{2})$, $(-\frac{1}{2}) \times (-\frac{2}{3}) = (\frac{1}{3})$, and $(-\frac{2}{3}) \times (-\frac{1}{2}) = (\frac{1}{3})$.

What are different ways a negative rational number can be interpreted?

M.P.7. Look for and make use of structure. Apply the properties of operations to rational numbers to determine the possible placements of a negative sign. For example, the number $-\frac{4}{17}$ is equivalent to $\frac{-4}{17}$ and $\frac{4}{-17}$. Additionally, demonstrate that a rational number with two negative signs can be written as an equivalent rational number without negative signs.

- Provide students with a fraction that includes a negative sign and ask them to write it in two other equivalent forms. For example, given the fraction $-\frac{17}{23}$, write $\frac{-17}{23}$ and $\frac{17}{-23}$.

- Provide students with a fraction that includes a negative sign and ask them to write an equivalent division expression. For example, given the fraction $\frac{19}{-3}$, write the expression $19 \div (-3)$.

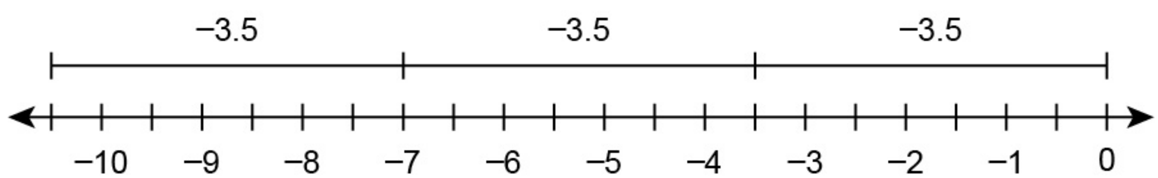
- Provide students with a list of fractions and ask them to identify all the fractions that are equivalent. For example, given the fractions $\frac{a}{b}$, $\frac{b}{a}$, $\frac{-a}{b}$, $\frac{a}{-b}$, $\frac{-a}{-b}$, $\frac{-b}{a}$, and $-\frac{a}{b}$, identify that $\frac{a}{b}$ and $\frac{-a}{-b}$ are equivalent and $\frac{-a}{b}$, $\frac{a}{-b}$, and $-\frac{a}{b}$ are equivalent.

How can real-world situations be used to interpret quotients of rational numbers?

M.P.2. Reason abstractly and quantitatively. Contextualize a division equation to interpret and make meaning out of negative and positive quotients. For example, the equation $(-16) \div 6 \approx -2.67$ could represent that a diver dove to a total depth of 16 feet below sea level in 6 minutes. As such, the diver dove an average of 2.67 feet per minute below sea level. Additionally, use number lines or other diagrams to illustrate and interpret the quotients of rational numbers in real-world contexts.

- Provide students with a real-world problem and ask them to write and explain a division equation that could be used to determine a solution. For example, Clare owes her neighbor $273 for a lawnmower that she purchased. Clare agrees to pay her neighbor an equal amount each month for 6 months. To determine the amount that Clare owes each month, evaluate the expression $-273 \div 6$. The quotient of -45.5 indicates that Clare owes her neighbor $45.50 each month for 6 months.

- Provide students with a real-world problem and ask them to solve it using a number line. For example, three friends owe $10.50 for a cheese pizza and want to share the cost equally. Using a number line, draw an arrow or line segment representing -10.50, then divide the segment into three equal sections. As such, each friend owes $3.50, which is equivalent to -3.5 on the number line.



How can the properties of operations be used to multiply and divide rational numbers?

M.P.2. Reason abstractly and quantitatively. Apply the properties of operations to problems involving rational numbers. For example, find the value of the expression $-\frac{3}{8} \times 4(6\frac{3}{4}) \times \frac{8}{3} \div \frac{1}{2}$ by first using the commutative property to rewrite it as $-\frac{3}{8} \times \frac{8}{3} \times 4(6\frac{3}{4}) \div \frac{1}{2}$. Then, use the multiplicative identity to note that $-\frac{3}{8} \times \frac{8}{3} = -1$. Use the distributive property to solve $4(6\frac{3}{4}) = 4(6 + \frac{3}{4}) = 24 + 3 = 27$. Finally, use the multiplicative inverse of $\frac{1}{2}$ to rewrite the whole expression as $-1 \times 27 \times 2$ for a solution of -54. Additionally, identify when problems that involve multiplying and dividing rational numbers can be solved using more than one strategy.

- Provide students with a list of steps showing how to evaluate an expression and ask them to justify some or all the steps. For example, give students the expression shown and the five steps used to evaluate it.

	Expression	Reason
Step 1:	$-\frac{1}{2} \times 7 \times (\frac{1}{2} \div \frac{1}{4})$	given
Step 2:	$-\frac{1}{2} \times 7 \times (\frac{1}{2} \times \frac{4}{1})$	?
Step 3:	$-\frac{1}{2} \times 7 \times 2$	multiply
Step 4:	$-\frac{1}{2} \times 2 \times 7$	?
Step 5:	-1×7	multiplicative inverse
Step 6:	-7	multiply

Students should justify the second step as “multiplying by the reciprocal” or something similar and the fourth step as “commutative property” or something similar.

- Provide students with an expression involving multiplication or division of rational numbers and ask them to identify two different strategies for evaluating it. For example, ask students to consider two different ways of evaluating $6(-\frac{1}{2}-\frac{1}{3})$. The difference of $-\frac{1}{2}$ and $\frac{1}{3}$ can be multiplied by 6 to obtain -5. Alternatively, the distributive property could first be applied, followed by calculating the difference of -3 and 2, which is also -5. In this case, students may choose the second method so that it is not necessary to find the difference of two fractions with different denominators.

Key Academic Terms:

rational number, integer, quotient, divisor, multiply, divide, fraction, properties of operations

Additional Resources:

- Lesson: A concrete introduction to the abstract concepts of integers and algebra using algebra tiles
- Activity: Drill rig
- Lesson: Multiplying and dividing with negatives
- Lesson: Division of integers
- Video: Two kinds of division
- Video: Relationship of division and fractions
- Lesson: Multiplying and dividing rational numbers

4g

Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
4. Apply and extend knowledge of operations of whole numbers, fractions, and decimals to add, subtract, multiply, and divide rational numbers including integers, signed fractions, and decimals. g. Convert a rational number to a decimal using long division, explaining that the decimal form of a rational number terminates or eventually repeats.

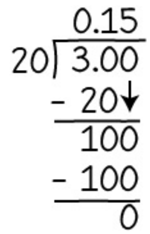
Guiding Questions with Connections to Mathematical Practices:

How can long division help to decide if a decimal form of a rational number repeats or terminates?

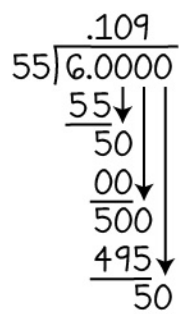
M.P.1. Make sense of problems and persevere in solving them. Divide the numerator of a fraction by its denominator to find its decimal form. For example, $\frac{5}{16}$ is equal to $5 \div 16$, or 0.3125, which does not repeat and terminates in the ten-thousandths place. By contrast, $\frac{7}{9}$ is equal to $7 \div 9$, or $0.777777\ldots$ with the digit 7 repeating, so the decimal can be written as $0.\overline{7}$. Additionally, use mental math and knowledge of place value to verify the results of long division.

- Provide students with the results of several long division problems and ask them to express the quotients using a bar called a vinculum. For example, when long division problems result in $0.676767\ldots$, $1.444444\ldots$, and $0.0121212\ldots$, write these results as $0.\overline{67}$, $1.\overline{4}$, and $0.0\overline{12}$.

- Provide students with a rational number and ask them to convert it to a decimal using long division. For example, $\frac{3}{20}$ can be converted as shown.



Because there is eventually a remainder of zero, the decimal representation of $\frac{3}{20}$ terminates. Then, ask students to convert $\frac{3}{20}$ to a fraction with a power of 10 in the denominator. In this case, $\frac{3}{20} \times \frac{5}{5} = \frac{15}{100}$. In this form, students can see that $\frac{3}{20}$ is equivalent to fifteen hundredths, which is the same as 0.15. On the other hand, consider the example of $\frac{6}{55}$. Use long division to convert the fraction as shown.



The second occurrence of a remainder of 5 indicates that the decimal representation will repeat rather than terminate. The first time it occurred, 50 divided by 55 gave a result of 0. This time, it will again give a result of 0 and will continue to repeat all the subsequent calculations, resulting in a quotient of 0.1090909. . . or $0.1\overline{09}$. Because the decimal does not terminate, it is not possible to convert $\frac{6}{55}$ to a fraction with a power of 10 in the denominator.

Key Academic Terms:

multiply, divide, rational, fraction, terminate, repeat, signed numbers

Additional Resources:

- Activity: Decimal expansions of fractions
- Lesson: Grade 7 mathematics module 2, topic B, lesson 14
- Tutorial: How do you turn a fraction into a terminating decimal?
- Tutorial: How do you turn a fraction into a repeating decimal?

5

Number Systems and Operations
Apply and extend prior knowledge of addition, subtraction, multiplication, and division to operations with rational numbers.
5. Solve real-world and mathematical problems involving the four operations of rational numbers, including complex fractions. Apply properties of operations as strategies where applicable.

Guiding Questions with Connections to Mathematical Practices:

How are rational numbers used to solve real-world and mathematical problems?

M.P.2. Reason abstractly and quantitatively. Represent situations with appropriate rational numbers and use operations suited to the given context. For example, to determine the total cost of an item, it may be necessary to multiply the price by a rational number that represents a discount (e.g., $\frac{1}{3}$ off means the price paid will be $\frac{2}{3}$ the original price). Additionally, identify when a real-world or mathematical problem involving rational numbers can be solved using different methods.

- Provide students with an original price or fee and ask them to write an expression that represents a fractional markup or decrease. For example, a plumber charges $85 per hour during the week. On weekends, she charges $\frac{1}{5}$ more per hour. Write the expression $(85 \times \frac{1}{5}) + 85$ to show an hourly weekend rate of $102.

- Provide students with a real-world problem involving rational numbers and ask them to solve it in multiple ways. For example, ask students to solve the following problem using addition only; addition and multiplication; and addition, multiplication, and the distributive property. Maria, Sam, and Oliver went to the movies. They each bought a ticket for $6.50, a drink for $4.00, and popcorn for $5.50. How much did they spend in total? Using addition only, evaluate $6\frac{1}{2} + 4 + 5\frac{1}{2} + 6\frac{1}{2} + 4 + 5\frac{1}{2} + 6\frac{1}{2} + 4 + 5\frac{1}{2}$. Using addition and multiplication, evaluate $3(6\frac{1}{2}) + 3(4) + 3(5\frac{1}{2})$. Using addition, multiplication, and the distributive property, evaluate $3(6\frac{1}{2} + 4 + 5\frac{1}{2})$ as $3(6\frac{1}{2}) + 3(4) + 3(5\frac{1}{2})$. All three methods produce a result of $48.

How can the rules for manipulating fractions be extended to help manipulate complex fractions?

M.P.6. Attend to precision. Observe that a complex fraction can be rewritten with a division sign instead of the fraction bar. For example, $\frac{\frac{-4}{7}}{\frac{1}{7}}$ can be rewritten as $\frac{-4}{7} \div \frac{1}{7}$. Additionally, know that changing a mixed number to an improper fraction can help solve or evaluate multiplication and division problems.

- Provide students with a complex fraction and ask them to create an equivalent expression using division signs and parentheses. For example, given the expression $\frac{\frac{7}{8}}{\frac{-1}{15}}$, write $(7 \div 8) \div (-1 \div 15)$.

- Provide students with a problem involving the quotient of two mixed numbers. Ask students to first write the problem as a complex fraction. Then, ask students to convert the mixed numbers to improper fractions and solve using multiplication. For example, Jake has $23\frac{1}{3}$ ounces of sunflower seeds. He divides the seeds equally into servings of $1\frac{2}{3}$ ounces. How many total servings does Jake have? The problem can be represented as $\frac{23\frac{1}{3}}{1\frac{2}{3}}$. When the mixed numbers are converted to improper fractions, the expression becomes $\frac{70}{3} \div \frac{5}{3}$. The value of the expression is equal to $\frac{70}{3} \times \frac{3}{5}$, which is $\frac{70}{5}$ or 14.

Key Academic Terms:

rational, operations, complex fractions, manipulate, signed numbers, division

Additional Resources:

- Activity: Sharing prize money
- Tutorial: Using rational numbers

6

Algebra and Functions
Create equivalent expressions using the properties of operations.
6. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

Guiding Questions with Connections to Mathematical Practices:

How does decomposing and composing linear expressions compare to decomposing and composing numeric expressions?

M.P.7. Look for and make use of structure. Use the properties of operations to compose and decompose linear expressions. For example, $4 - 6(2 - 7x)$ can be thought of as $4 + (-6) \bullet 2 + (-6) \bullet (-7x)$ or $-8 + 42x$. Additionally, know that when decomposing and composing linear expressions, the order of operations should be used in conjunction with the commutative, associative, and distributive properties to generate equivalent expressions. Further, illustrate subtracting as equivalent to adding the additive inverse and use this to apply the commutative and associative properties to subtraction.

- Ask students to generate an equivalent expression for a given expression by using the order of operations in conjunction with the commutative, associative, and distributive properties. For example, give students the expression $-5 - 3(2x - 6)$. The given expression contains a grouping symbol, but there are no operations within the grouping symbol that can be performed, as $2x$ and -6 are not like terms. The first operation that can be performed is multiplication, as there are no exponents. Rewrite the expression using addition of the opposite in place of subtraction. Then, multiply -3 by both $2x$ and -6.

$$-5 - 3(2x - 6)$$

$$-5 + (-3) \bullet (2x + (-6))$$

$$-5 + (-3) \bullet 2x + (-3) \bullet (-6)$$

$$-5 + (-6x) + 18$$

The commutative property allows the expression to be rewritten so that the like terms are first and the associative property allows that addition to be grouped so that it is performed first.

$$-5 + (-6x) + 18$$

$$-5 + 18 + (-6x)$$

Add –5 and 18 to get 13 and write the expression as $13 - 6x$, $13 + (-6x)$, or $-6x + 13$.

- Ask students to analyze and correct common mistakes that arise when generating equivalent expressions. For example, give students the prompt "A student generated the expression $-30x - 21$ and claims that it is equivalent to the expression $4 - 5(6x - 5)$. Explain the student's error and give the correct answer." Write the expression using addition of the opposite in place of subtraction so that the expression is $4 + (-5) \bullet (6x + (-5))$. Then, perform the multiplication by –5 to generate the expression $4 + (-30x) + 25$. Identify that the student's error was multiplying $5 \bullet (-5)$ rather than $(-5) \bullet (-5)$ when using the distributive property. Use the commutative property to generate the equivalent expression $4 + 25 + (-30x)$ and add the like terms to write the correct expression as $29 + (-30x)$, $29 - 30x$, or $-30x + 29$.

M.P.6. Attend to precision. Identify conventions in linear expressions, especially when dealing with negative numbers, and be aware of common misconceptions. For example, $4 - 6(2 - 7x)$ is not equivalent to $-2(2 - 7x)$. Additionally, know that the distributive property represents multiplication and should be performed where multiplication falls in the order of operations in order to generate an equivalent expression.

- Ask students to explain why the distributive property is used before like terms can be combined in expressions like $12 - 3(5 - 9x)$. For example, give students the expression $9 - 4(7 - 3x)$ and ask them to determine the first operation to perform on the expression and justify their response. The order of operations still applies when working with algebraic expressions. There is an operation inside the grouping symbol, but the terms 7 and $-3x$ cannot be combined. There are no exponents, so the first operation to perform is multiplication because the distributive property is simply the multiplication of a factor containing more than one term. Additionally, the correct value to use in the distributive property for the given problem is –4. This is because if only 4 is used, the expression becomes $9 - (28 - 12x)$. In order to multiply so that there are no longer parentheses, the subtraction should be treated as addition of the opposite, and, therefore, –4 is distributed to both 7 and $-3x$.

- Ask students to explain how the commutative and associative properties can be used with subtraction through the use of integers. For example, give students the expression $-6 - 4x + 9 - 7x$. Generate the equivalent expression $-6 + (-4x) + 9 + (-7x)$ by representing the subtraction as addition of the opposite. The expression is now represented entirely using addition, so the commutative property can be applied. This leads to the conclusion that the commutative property can be applied to all problems containing subtraction as long as the subtraction is represented through addition of the opposite. Generate the equivalent expression $(-4x) + (-7x) + (-6) + 9$ using the commutative property and then use the associative property to group the terms containing x and the constant terms so that the like terms can be added.

$$-6 + (-4x) + 9 + (-7x)$$

$$(-4x) + (-7x) + (-6) + 9$$

$$((-4x) + (-7x)) + ((-6) + 9))$$

$$-11x + 3$$

The expression can be represented as $-11x + 3$, $3 - 11x$, or $3 + (-11x)$.

Key Academic Terms:

properties of operations, factor, expand, equivalent, expression, compose, decompose, coefficient, like terms, term, constant

Additional Resources:

- Activity: Writing expressions
- Lesson: Expand a linear expression with rational coefficients
- Video: Commutative and associative properties of addition
- Lesson: Factor a linear expression with rational coefficients
- Activity: Ticket to ride
- Tutorial: How do you distribute with whole numbers and fractions?
- Tutorial: How do you factor a greatest common factor out of an expression?

7

Algebra and Functions
Create equivalent expressions using the properties of operations.
7. Generate expressions in equivalent forms based on context and explain how the quantities are related.

Guiding Questions with Connections to Mathematical Practices:

How can different forms of an expression help conceptualize problems in a different way?

M.P.7. Look for and make use of structure. Apply the properties of operations to expressions to see problems in a different way. For example, a 20% off sale of everything in a store can be represented by multiplying the price, p, by 0.2 and then subtracting that result from the original price: $p - 0.2p$. Another form of that expression is $0.8p$, which highlights that 20% off can also be seen as 80% of the original price. Additionally, know that a variable term that does not contain a coefficient, such as p, is understood to have a coefficient of 1. Further, know that a variable term that does not contain a coefficient can be considered 100% of the value of the variable.

- Ask students to generate equivalent expressions in order to help conceptualize the situation that an expression is used to represent. For example, give students the expression $0.85x$ and tell students that the expression represents the current cost of a shirt after a discount has been applied to the original price, x. Generate the equivalent expression $x - 0.15x$ to explain the discount percentage that has been applied. This expression is used because it represents the original price x being lowered by subtracting a multiple of x. The multiple of x that is being subtracted is $0.15x$, meaning the discount is 0.15 of the original price, which can be represented as 15%. The price of the shirt has therefore been discounted by 15%.

- Ask students to construct a pattern that explains how coefficients, percent increase, and percent decrease are related. An example is shown.
    - Charlotte purchases a new pair of headphones. The price of the headphones can be represented by the variable h. The sales tax on the headphones is 8% of the price. Write an expression to represent the total amount Charlotte must pay based on the price of the headphones, h.

      I can represent the cost of the headphones as h and the amount of the sales tax as $0.08h$ since 8% is the same as 0.08 and a percentage of a cost can be calculated by multiplying the numeric value of the percentage by the cost. The expression $h + 0.08h$ can be used to represent the total cost of the headphones with the tax. This expression is equivalent to $1.08h$ because the coefficients can be added.

  Additionally, give students the following situation:

    - Victor is purchasing a toy for his younger brother. The original price of the toy can be represented by the variable t. The toy has a discount of 10% because the box has been opened. Write an expression to represent the cost of the toy after the discount.

      I can represent the cost of the toy as t and the amount of the discount as $0.1t$ since 10% is the same as 0.1 and a percentage of a cost can be calculated by multiplying the numeric value of the percentage by the cost. The expression $t - 0.1t$ can be used to represent the cost of the toy after the discount. This expression is equivalent to $0.9t$ because the coefficients can be subtracted.

  Generalize these examples to demonstrate an expression ax represents an amount after a percent increase when $a > 1$, or a percent decrease when $a < 1$. Find the absolute value of the difference between a and 1 to find the percentage by which the original amount has been increased or decreased.

Key Academic Terms:

properties of operations, equivalent, expression

Additional Resources:

- Activity: Ticket to ride
- Video: Equivalent expressions with the distributive property

8

Algebra and Functions
Solve real-world and mathematical problems using numerical and algebraic expressions, equations, and inequalities.
8. Solve multi-step real-world and mathematical problems involving rational numbers (integers, signed fractions and decimals), converting between forms as needed. Assess the reasonableness of answers using mental computation and estimation strategies.

Guiding Questions with Connections to Mathematical Practices:

How can the properties of operations be used to solve multistep real-life and mathematical problems involving positive and negative rational numbers?

M.P.7. Look for and make use of structure. Represent real-life situations mathematically using variables for unknowns and apply the properties of operations to solve for the unknowns. For example, "Jonas is saving for a new bike that costs \$100 plus \$7.50 in sales tax. If he already has \$20, how much more does Jonas need to save for the new bike?" This situation can be represented by the equation $100 + 7.5 - 20 = m$, where m is the amount of money Jonas still needs to save. Additionally, verify that inverse operations can be used to solve problems.

- Ask students to represent real-life situations using numbers, operations, and variables. For example, give students the situation "Anika bought a new pair of shoes. The shoes were on sale for 10% off, and Anika had a \$5-off coupon. The coupon was applied before the percent discount. Anika then had to pay \$7.50 in shipping. The total cost of the shoes was \$59.25. What was the original price of the shoes?" Represent the unknown quantity, which is the original price of the shoes, using a variable, p. The shipping was added last, so the total cost of \$59.25 will decrease by \$7.50 to undo the addition. The percent discount was applied before the shipping cost, so $(59.25 - 7.5)$ must be divided by 0.9, as the original price minus the coupon was multiplied by 0.9 to find the cost after the sale. The expression $(59.25 - 7.5) \div 0.9$ must then be increased by \$5, as the coupon subtracted \$5 from the original price. The original price can therefore be represented as $p = (59.25 - 7.5) \div 0.9 + 5$.

- Ask students to solve real-life problems that contain positive and negative rational numbers using inverse operations. For example, give students the situation “Cody is baking $2\frac{1}{2}$ batches of cornbread muffins. The recipe calls for $1\frac{1}{2}$ cups of flour per batch. Cody does not have enough flour and will have to borrow $1\frac{1}{4}$ cups of flour from Ben. How much flour does Cody have before he borrows some flour from Ben?” Determine that the total amount of flour that Cody needs to make $2\frac{1}{2}$ batches of muffins is $3\frac{3}{4}$ cups, because the product of $2\frac{1}{2}$ and $1\frac{1}{2}$ is $3\frac{3}{4}$. The product is used because the phrase “each batch” indicates that the number of batches should be multiplied by the amount per batch. The product can be found by decomposing $2\frac{1}{2}$ into $2 + \frac{1}{2}$ and $1\frac{1}{2}$ into $1 + \frac{1}{2}$, multiplying each part of one number by each part of the other, and adding the products together so that $(2 \times 1) + (2 \times \frac{1}{2}) + (1 \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2}) = 3\frac{3}{4}$. The amount of flour that Cody has can then be found by subtracting the amount he needs to borrow from the total amount he needs, which means Cody already has $3\frac{3}{4} - 1\frac{1}{4}$ or $2\frac{1}{2}$ cups of flour.

M.P.1. Make sense of problems and persevere in solving them. Convert between different forms of rational numbers to solve problems efficiently. For example, write $\frac{4}{7} \times 0.2 - 3.4$ as $\frac{4}{7} \times \frac{2}{10} - \frac{34}{10}$. Additionally, identify when a specific form of a rational number is the most advantageous for solving a particular problem. For example, if an exact solution is required and the decimal form of the answer results in a repeating decimal, a fraction may be the best form of the number to use.

- Ask students to solve problems that can be solved more efficiently by converting between different forms of rational numbers. For example, ask students to find an equivalent value for $\frac{2}{9} \times 2.5 - 5\frac{3}{4}$. Determine that 2.5 can be represented as $\frac{5}{2}$, and since the other terms in the expression are represented as fractions, it would be best to also express 2.5 in its fractional form of $\frac{5}{2}$. Perform the multiplication to get $\frac{2}{9} \times \frac{5}{2} = \frac{10}{18} = \frac{5}{9}$ and generate the equivalent expression $\frac{5}{9} - 5\frac{3}{4}$. To easily perform the subtraction, rewrite the expression using equivalent fractions that have a common denominator. Then, determine that $\frac{5}{9}$ can be represented as $\frac{20}{36}$ and $5\frac{3}{4}$ can be represented as $\frac{23}{4}$, or $\frac{207}{36}$, and perform the subtraction to determine that the value of the expression is $-\frac{187}{36}$, or $-5\frac{7}{36}$.

- Ask students to justify why a particular form of a rational number makes solving a particular problem more efficient than another form. For example, ask students to explain why using the fraction forms of both numbers in the expression $0.5 \times \frac{1}{4}$ is the most efficient method to determine the value of the expression. Explain that when both numbers are represented as fractions, the multiplication to be performed is $\frac{1}{2} \times \frac{1}{4}$, which can be easily calculated by simply multiplying the numerators together and the denominators together to get $\frac{1}{8}$. If the decimal forms of both numbers were used instead, the problem becomes 0.5×0.25, which, in using the standard algorithm, requires multiplication of a double-digit number by a single digit in addition to determining where the decimal point is placed in the answer. Additionally, ask students to explain why using the decimal forms of both numbers in the expression $0.365 + \frac{1}{4}$ is the most efficient method to determine the value of the expression. Explain that when both numbers are represented as decimals, the addition to be performed is $0.365 + 0.25$, which can be easily carried out by adding the thousandths, hundredths, and tenths digits of each number to get 0.615. If the fraction forms of both numbers were used instead, 0.365 would need to be represented as $\frac{365}{1{,}000}$, or $\frac{73}{200}$. Then, a common denominator of 200 would need to be used to be able to add the fractions.

How can mental computation and estimation strategies help to assess the reasonableness of an answer?

M.P.3. Construct viable arguments and critique the reasoning of others. Compare an actual computation to an estimate that uses appropriate place value to assess the reasonableness of the computation. For example, estimate 15.4×3.7 as 15×4 to ensure the decimals are placed appropriately in the final computation. Additionally, know that relative place value is important when comparing an actual computation to an estimate. For example, an estimated value of 900 miles is relatively close to an exact answer of 905.2 miles, but an estimated value of 5 inches is not relatively close to an exact answer of $8\frac{2}{3}$ inches.

- Ask students to determine an appropriate estimate to assess the reasonableness of an answer. For example, ask students to determine an appropriate estimate for 24.8×30.4. Determine that 24.8 is relatively close to (but slightly less than) 25 and 30.4 is relatively close to (but slightly more than) 30. The values of 25 and 30 are selected because numbers that contain a digit of 0 or 5 in the ones place make mental computation more feasible. Determine that 25×30 is 750 by decomposing 25 into $20 + 5$ and multiplying 20 by 30 and 5 by 30, then adding the products together. Then, compare the estimate to the exact answer of 753.92 and explain that the numbers are relatively close when considering the fact that the digits in the hundreds and tens place are identical. Conclude that the estimated value is a good estimate for the expression, and the computed solution is reasonable because of the relatively close value of the solution when compared with the estimated value.

- Ask students to use estimation strategies to analyze errors made in computation. For example, give students the prompt "A student evaluated the expression 38.1×15.8 and got the incorrect answer 6,019.8. Without evaluating the expression, use an estimation strategy to explain why the value the student got is not reasonable." Determine that 38.1 can be approximated by 40 and 15.8 can by approximated by 15. Multiply 40 by 15 by decomposing 15 into $10 + 5$ and multiplying 10 by 40 and 5 by 40, then adding the products 400 and 200 together to get 600. The solution the student got was about 6,000, so the student likely made an error involving place value as the answer given by the student is not reasonable based on the estimate of 600.

Key Academic Terms:

properties of operations, equivalent, expression, estimation, place value

Additional Resources:

- Activity: Gotham city taxis
- Activity: Stained glass

9a

Algebra and Functions
Solve real-world and mathematical problems using numerical and algebraic expressions, equations, and inequalities.
9. Use variables to represent quantities in real-world or mathematical problems and construct algebraic expressions, equations, and inequalities to solve problems by reasoning about the quantities. a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach.

Guiding Questions with Connections to Mathematical Practices:

How can using variables help to represent problems and find solutions?

M.P.7. Look for and make use of structure. Represent any unknown quantity with a variable and interpret the situation mathematically. For example, if the sum of four consecutive integers is 46, then the variable c can be used to represent the first integer and $(c + 1)$, $(c + 2)$, and $(c + 3)$ can be used to represent the subsequent integers such that $c + (c + 1) + (c + 2) + (c + 3) = 46$, or $4c + 6 = 46$. Additionally, illustrate that equivalent forms of an equation generated to represent a situation may be more efficient in determining the value of the unknown quantity, while the original form may be more useful in explaining how the equation matches the situation it represents.

- Ask students to represent a situation containing an unknown quantity using an equation that uses a variable to represent the unknown quantity. For example, give students the situation "On Monday, Ethan read for a certain number of minutes. Each day during the week, he increased the number of minutes he read that day by 5 from the previous day. By Friday, Ethan had read a total of 210 minutes. How many minutes did Ethan read on Monday?" Determine that the unknown quantity is the number of minutes Ethan read on Monday and use a variable, m, to represent this quantity. The number of minutes Ethan read on Tuesday can therefore be represented as $m + 5$. The number of minutes Ethan read on Wednesday is 5 more than $m + 5$, which is $m + 10$. The number of minutes Ethan read on Thursday is therefore $m + 15$, and the number of minutes Ethan read on Friday can be represented by $m + 20$. The problem states that the total number of minutes is 210, which indicates that the sum of these numbers is 210. The situation can therefore be represented as $m + (m + 5) + (m + 10) + (m + 15) + (m + 20) = 210$. The left side of the equation contains only addition, so the associative property can be used to eliminate the parentheses, and the commutative property can be used in conjunction with the associative property to regroup the variable terms together and the constants together to generate the equivalent equation $5m + 50 = 210$.

$$\begin{aligned} m + (m + 5) + (m + 10) + (m + 15) + (m + 20) &= 210 \\ m + m + 5 + m + 10 + m + 15 + m + 20 &= 210 \\ (m + m + m + m + m) + (5 + 10 + 15 + 20) &= 210 \\ 5m + 50 &= 210 \end{aligned}$$

  In this situation, the form $m + (m + 5) + (m + 10) + (m + 15) + (m + 20) = 210$ more directly corresponds with the context, however simplifying to the form $5m + 50 = 210$ makes the problem much easier to solve.

- Ask students to solve problems containing an unknown quantity either by writing and solving an equation to represent the situation or by applying the procedure of solving the equation within the context given. For example, give students the situation "Maya is buying a set of building blocks. The large package contains 50% more blocks than the small package. The large package contains 132 blocks. How many blocks does the small package contain?" Determine that when a quantity is increased by 50%, the new quantity is 150% of the original quantity, and 150% of a number can be determined by multiplying the number by 1.5. Generate the equation $1.5b = 132$ where b represents the number of blocks in the small package and determine that the equation can be solved by dividing both sides of the equation by 1.5 to generate $b = 88$. Conclude that the small package of building blocks contains 88 blocks.

$$\frac{1.5b}{1.5} = \frac{132}{1.5}$$
$$b = 88$$

How does an arithmetic solution compare to an algebraic solution?

M.P.1. Make sense of problems and persevere in solving them. Observe that algebraic solutions and arithmetic solutions result in the same answer using related strategies, and determine when to use one over the other. For example, take the following scenario: "Avery had $50 and then he purchased 3 movie tickets for $14 each." The amount of money remaining can be represented as $50 - 14 - 14 - 14$ using arithmetic or $3 \bullet 14 + x = 50$ algebraically, where x is the amount of money remaining. Additionally, observe that when algebraic expressions make up part of an equation, these expressions can be decomposed to demonstrate that the algebraic solution matches the arithmetic solution. Further, observe that repeated addition or subtraction can be represented using multiplication, which can lead to a more efficient algebraic or arithmetic solution.

- Ask students to determine arithmetic and algebraic solutions to real-world problems. For example, give students the following prompt: “Mai needed to read 140 pages of a book. She read 25 pages each day for 4 days. How many pages does Mai have left to read?” Determine that an arithmetic solution to the situation can be found by starting with the amount Mai initially had to read and subtracting the amount she reads each day to find the amount she still needs to read. This expression is 140 – 25 – 25 – 25 – 25, because Mai initially needed to read 140 pages and read 25 pages each day for 4 days, so the total number of pages Mai still needs to read is the original amount reduced by 25 four times. Conclude that Mai still needs to read 40 pages. Additionally, determine that the same situation can be represented algebraically. The number of pages Mai needed to read is 140. The number of pages she has read is 25 pages for each of 4 days, which can be represented by the expression 4 • 25. The number of pages she has left to read so that her total number of pages is 140 is an unknown quantity, p. Therefore, the expression $4 \bullet 25 + p$ represents the number of pages Mai will have read in total, and since the problem states that this must be equivalent to 140, the equation $4 \bullet 25 + p = 140$ can be used to represent the situation. The equation can be solved by subtracting 100 from both sides to generate the equivalent equation $p = 40$, which is the same as the arithmetic solution.

- Ask students to distinguish between situations for which an arithmetic solution is more efficient and appropriate and those for which an algebraic solution is more efficient and appropriate. For example, give students the situation “Aaron is saving for a new phone. The phone costs \$550. Aaron earns and saves \$50 per week by mowing lawns. He must first pay his uncle back for the \$100 he used to buy the lawn mower. After how many weeks will Aaron have saved enough to buy the phone after having paid his uncle back?” Determine that the arithmetic solution $(550 + 100) \div 50$ is an efficient way to solve the problem because the solution can be determined by adding the cost of the phone to the amount Aaron owes and then finding the number of weeks that it will take Aaron to have \$650. These operations can be done mentally by determining that it takes two weeks to save \$100, so it would take 12 weeks to save \$600, and the final \$50 will be saved in week 13.

How can the properties of operations and the structure of an equation help solve equations flexibly and efficiently?

M.P.2. Reason abstractly and quantitatively. Analyze a problem to determine an efficient strategy for solving it. For example, when given the problem $3(x-2)=15$, solve by first dividing the equation by 3, rather than using the distributive property. Additionally, observe that when solving an equation containing a fractional coefficient on a variable, multiplying by the reciprocal is more efficient than dividing by the value itself. For example, when solving $\frac{2}{7}x=6$, multiplying by $\frac{7}{2}$ will isolate the variable and is more efficient than dividing by $\frac{2}{7}$, resulting in $x=6\times\frac{7}{2}=\frac{42}{2}=21$.

- Ask students to compare strategies to solve an equation and determine which strategy is most efficient for a given equation. For example, give students the equation $-2(w+4)=-14$. Determine that the equation can be solved by either using the distributive property to generate an equivalent expression for the left side of the equation and then isolating the variable or by first dividing by –2 on both sides of the equation. Solve the equation using the first method by generating the equivalent expression $-2w+(-8)$ and writing the equivalent equation $-2w+(-8)=-14$. Add 8 to both sides of the equation to generate the equivalent equation $-2w=-6$ and then divide both sides of the equation by –2 to get the solution $w=3$.

$$\begin{array}{r}
-2(w+4) = -14 \\
-2w + (-8) = -14 \\
\underline{\qquad +8 \quad\; +8} \\
\dfrac{-2w}{-2} = \dfrac{-6}{-2} \\
w = 3
\end{array}$$

Solve the equation using the second method by first dividing both sides of the equation by –2. This operation can be performed because the parentheses indicate that the entire expression inside the parentheses on the left side of the equation is being multiplied by –2, so dividing will leave only the expression inside the parentheses. This generates the equivalent equation $w + 4 = 7$, which can be solved by subtracting 4 from both sides to yield the solution $w = 3$.

$$\frac{-2(w+4)}{-2} = \frac{-14}{-2}$$
$$w + 4 = 7$$
$$-4 \quad -4$$
$$w = 3$$

Determine that for this equation, the second method is the most efficient because, while the distributive property can be used on the left side of the equation, it is not necessary and creates extra steps in the procedure that do not need to be performed.

- Ask students to solve an equation containing a fractional coefficient on the variable by multiplying by the reciprocal of the coefficient and explaining why this is more efficient than division. For example, give students the equation $\frac{3}{8}x + 9 = -12$. Determine that the equation can be solved by first subtracting 9 from both sides of the equation. This yields the equivalent equation $\frac{3}{8}x = -21$.

$$\frac{3}{8}x + 9 = -12$$
$$-9 \quad -9$$
$$\frac{3}{8}x = -21$$

To solve the equation, a value for x needs to be determined. The variable x is the same as $1x$, and the left side of the equation becomes $1x$ when $\frac{3}{8}$ is multiplied by its reciprocal, $\frac{8}{3}$. Solve the equation by multiplying both sides of the equation by $\frac{8}{3}$, which gives the solution $1x = -56$, or $x = -56$.

$$\frac{8}{3} \bullet \frac{3}{8}x = \frac{-21}{1} \bullet \frac{8}{3}$$
$$1x = -56$$
$$x = -56$$

Key Academic Terms:

properties of operations, equivalent, expression, algebraic, arithmetic, variable, inequality, properties of equality

Additional Resources:

- Lesson: A concrete introduction to the abstract concepts of integers and algebra using algebra tiles
- Video: Solving linear equations with negative numbers
- Article: How to teach solving equations

9b

Algebra and Functions
Solve real-world and mathematical problems using numerical and algebraic expressions, equations, and inequalities.
9. Use variables to represent quantities in real-world or mathematical problems and construct algebraic expressions, equations, and inequalities to solve problems by reasoning about the quantities. b. Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality, and interpret it in the context of the problem.

Guiding Questions with Connections to Mathematical Practices:

How does an inequality compare to an equation?

M.P.2. Reason abstractly and quantitatively. Observe that with an equation, one expression is set equal to another expression using an equal sign ($=$). With an inequality, one expression is set less than ($<$), greater than ($>$), less than or equal to ($\leq$), or greater than or equal to ($\geq$) another expression. For example, the word problem "Marcy saves $5.00 per day, *x*, and already has $2.00. She wants to save $17.00," can be represented by the equation $5x + 2 = 17$. The word problem "Jonas saves $5.00 per day, *x*, and already has $2.00. He wants to save at least $17.00," can be represented by the inequality $5x + 2 \geq 17$. Additionally, observe that key phrases indicate which symbol to use. For example, the phrase "at least," indicates that the $\geq$ symbol should be used and the phrase "no more than" indicates that the $\leq$ symbol should be used.

- Ask students to compare situations involving equations and situations involving inequalities. For example, give students the situation “Paul practiced his trumpet for 34 minutes on Tuesday, which was 8 minutes less than twice as long as he practiced on Monday.” Determine that the situation is best represented using an equation because “was” is a statement of equality. Words and phrases like “was,” “is,” or “is the same as” indicate that two quantities are equivalent and can be compared using the equal sign. Additionally, give students the situation “Vanessa needed to read at least 124 pages of a book. She has already read 32 pages and will read 20 pages per day.” The situation is best represented using an inequality because the situation states that Vanessa must read “at least” 124 pages. The use of the phrase “at least” indicates that two quantities are either equal or that the first quantity is greater than the second quantity. Explain that key phrases like “at least,” “no more than,” “more than,” or “less than” often indicate that two quantities can be related using an inequality symbol such as $\geq$, $\leq$, $>$, or $<$.

- Ask students to represent situations algebraically using inequalities. For example, give students the situation “Jack would like to spend no more than $35 on snacks for a party. He spent $20 on crackers and will spend the rest on juice. A bottle of juice costs $2. How many bottles of juice can Jack buy?” The amount Jack spends on crackers and juice can be represented by the expression $2b + 20$, where b is the number of bottles of juice that Jack buys. The phrase “no more than” indicates that $2b + 20$ is not more than $35, i.e., it is less than $35 or it is equal to $35. Therefore, the inequality $2b + 20 \leq 35$ can be used to represent the situation. Observe that the sides of the inequality may be reversed so long as the direction of the sign stays with its value. The inequality $2b + 20 \leq 35$ is equivalent to $35 \geq 2b + 20$.

- Ask students to extend principles of equations and apply those principles to inequalities. For example, ask students to determine whether the subtraction property of equality applies to inequalities. Give students an inequality such as $5 > 2$ and ask them to determine whether subtracting the same number from both quantities preserves the inequality. Some possible student work is shown.

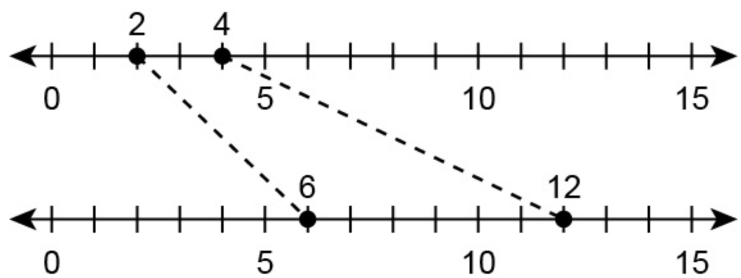
$$\begin{array}{ccc} 5 > 2 & 5 > 2 & 5 > 2 \\ 5 - 1 \overset{?}{>} 2 - 1 & 5 - \frac{2}{3} \overset{?}{>} 2 - \frac{2}{3} & 5 - 8 \overset{?}{>} 2 - 8 \\ 4 > 1 & 4\frac{1}{3} > 1\frac{1}{3} & -3 > -6 \end{array}$$

Conclude that the subtraction property does seem to apply to inequalities as well as equations. As a further example, ask students to determine whether the multiplication property of equality applies to inequalities. Give students an inequality such as $6 < 8$ and ask them to determine whether multiplying both quantities by the same number preserves the inequality. Some possible student work is shown.

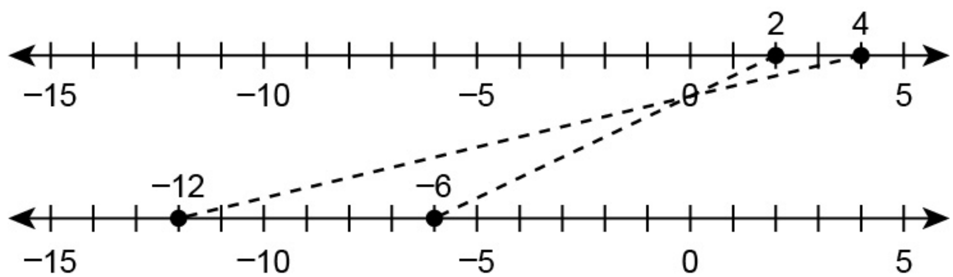
$$\begin{array}{ccc} 6 < 8 & 6 < 8 & 6 < 8 \\ 6 \bullet 3 \overset{?}{<} 8 \bullet 3 & 6 \bullet \frac{1}{2} \overset{?}{<} 8 \bullet \frac{1}{2} & 6 \bullet -2 \overset{?}{<} 8 \bullet -2 \\ 18 < 24 & 3 < 4 & -12 > -16 \end{array}$$

Observe that because the ordering of negative numbers is the opposite of positive numbers, the multiplication property preserves the inequality only when multiplying by positive numbers and reverses the inequality when the multiplier is negative.

- Ask students to use number lines to compare inequalities that involve multiplying by negative numbers. For example, locate 2 and 4 on a number line, noting that $2 < 4$. Then, multiply 2 and 4 each by 3, and map the two products on a new number line. Note that the mapping lines do not cross, and that just as $2 < 4$, $2 \bullet 3 < 4 \bullet 3$, since $6 < 12$.



Next, follow the same process, but this time multiply 2 and 4 each by –3.

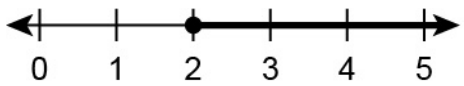
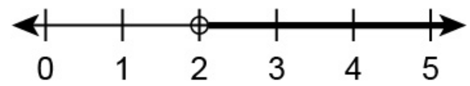


Note that the mapping lines now cross, changing the inequality. So, $2 \bullet -3 > 4 \bullet -3$, because $-6 > -12$. Whenever an inequality is multiplied by a negative number, the sign of the inequality reverses.

How is the solution set of an inequality represented?

M.P.4. Model with mathematics. Represent more than one solution, including infinitely many solutions, with a graph and interpret its meaning within the context of the problem. For example, the solution set to the number of hours a person needs to work to earn at least $75.60 if she makes $14 an hour and pays 10% in income tax can be represented with the inequality $14h - 1.4h \geq 75.6$, where h is the number of hours. A number line with a closed point plotted at 6 and an arrow pointing to the right shows that she needs to work 6 hours or more to meet her goal. Additionally, know that a point on a graph that is closed indicates that the point is included in the solution set, while a point on a graph that is open indicates that the point is a boundary of the solution set but not a solution itself. Further, know that the solution set of an inequality often contains an infinite number of values and therefore cannot be represented using a list.

- Ask students to explain the difference between a graph with a closed point and an open point. For example, ask students to explain the difference in the meanings of the following graphs.

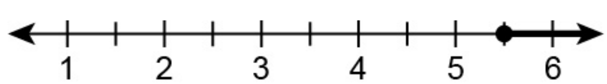



Determine that while both graphs include all values greater than 2 in their solution sets, the graph on the left includes 2 in the solution set, and the graph on the right does not. Explain that the open circle indicates that not only is 2 not in the solution set, but that the solution set includes points that go as close to 2 as needed, including 2.1, 2.01, and 2.001.

- Ask students to represent the solution set of a situation by writing an inequality for the situation and graphing the solution set on a number line. For example, give students the situation "Kayla is training for a bike race. Kayla wants to bike at least 132.5 miles per week. Kayla has already biked 50 miles this week and can ride at a rate of 15 miles per hour. Write an inequality that can be used to find the number of hours that Kayla must bike for the rest of the week to meet her goal. Show the solution set on a number line." The expression $15h + 50$ represents the number of miles Kayla will have biked after h additional hours of riding. The phrase "at least" indicates that this amount must be greater than or equal to 132.5. Therefore, the inequality $15h + 50 \geq 132.5$ can be used to represent the situation.

$$\begin{array}{r} 15h + 50 \geq 132.5 \\ -50 \quad\; -50 \\ \hline \dfrac{15h}{15} \geq \dfrac{82.5}{15} \\ h \geq 5.5 \end{array}$$

Show the solution set on a number line by plotting a closed endpoint at 5.5 since the value 5.5 is included in the solution set. Then, draw an arrow to the right of 5.5 to indicate that all values greater than 5.5 are included in the solution set of the inequality.



What is the meaning of a solution set in an inequality?

M.P.6. Attend to precision. Interpret a real-world solution set of an inequality. For example, given the context “The lowest recorded temperature was 5°F,” the solution set would be temperatures greater than or equal to 5°F. Additionally, interpret individual values within the solution set.

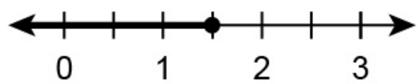
- Ask students to explain what it means for a number to be in the solution set of an inequality. For example, give students the inequality $2x + 5 < 11$. The solution set is $x < 3$. This means that all numbers in the solution set will create a true comparison when substituted into the original inequality, and all numbers not in the solution set will create a false statement when substituted into the original inequality.

- Ask students to interpret the numbers in the solution set of a problem in context. For example, give students the situation “Celeste’s plant is currently 12 inches tall and grows 2 inches every week. When will her plant be over 30 inches tall?” Represent the situation using the inequality $2w + 12 > 30$ where w is the number of weeks from now. The solution set of this inequality is $w > 9$. Each value in the solution set is a number of weeks after which the height of Celeste’s plant will be taller than 30 inches, based on the given information. Therefore, her plant will be taller than 30 inches after 9 weeks.

How does the context of a problem affect the reasonableness of the solution set in an inequality?

M.P.6. Attend to precision. Consider the solution set in the context of the problem and determine whether all values in the solution set are meaningful. For example, the number of school buses needed for a field trip is at least two, which is modeled on a number line with a closed point at 2 and an arrow pointing right, showing all values 2 and greater. In the context of the problem, the solution set only includes whole numbers of school buses. Additionally, there is an upper limit to the solution because the number of buses does not increase without bound. Further, there are many contexts in which negative solutions are not meaningful.

- Ask students to interpret the solution set of an inequality in which a lower limit is implied by the context. For example, give students the situation "George's parents tell him that he can watch no more than an hour and a half of television per day." Represent the situation using the inequality $h \leq 1\frac{1}{2}$ where h is the number of hours of television that George watches in one day. The solution set of the inequality is shown.



Explain that while the solution set of the algebraic inequality does not have a specified lower limit, the context dictates that there is a lower limit of 0 because it is not possible to watch a negative amount of hours of television. As an extension, explain that the lower limit of 0 can be included in the algebraic inequality by writing it as $0 \leq h \leq 1\frac{1}{2}$. The new solution set can be represented on a number line by plotting closed endpoints at both 0 and $1\frac{1}{2}$. All values between the endpoints are included in the solution set.

- Ask students to interpret the solution set of an inequality where the context implies only whole number solutions. For example, give students the situation "Grace is taking a 50-question exam as part of a lifeguard certification course. She needs to answer at least 40 of the questions correctly to pass the exam. There is no partial credit. Write an inequality to represent this situation." Determine that the inequality $40 \leq q$ can be used to represent the number of questions Grace must answer correctly to pass her exam, and the inequality $q \leq 50$ represents the limit of 50 possible questions on the entire exam. Grace will either answer each question correctly or incorrectly, so her score can only include whole numbers. Therefore, decimal values are solutions to the algebraic representation, but are not solutions in the given context. The solution set of the algebraic representation contains an infinite number of values between 40 and 50, inclusive, while the solution set of the context contains only the whole numbers from 40 to 50.

Key Academic Terms:

properties of operations, equivalent, expression, inequality, greater than (>), less than (<), greater than or equal to (≥), less than or equal to (≤), solution set, properties of equality

Additional Resources:

- Activity: Fishing adventures 2
- Activity: Sports equipment set
- Video: Piano tuning and balancing equations

10a

Data Analysis, Statistics, and Probability
Make inferences about a population using random sampling.
10. Examine a sample of a population to generalize information about the population. a. Differentiate between a sample and a population.

Guiding Questions with Connections to Mathematical Practices:

How can a sample of a population help to understand the population statistically?

M.P.2. Reason abstractly and quantitatively. Observe a small group of a population, called a sample, to make generalizations about the population as a whole, which is especially useful if many samples are taken and observed. For example, instead of asking the 70,000 people in attendance at a professional football game a survey question, take a representative sample of that population by selecting 100 people to survey and use that information to make generalizations about the entire population. Additionally, know that the conditions under which the sample is surveyed should remain constant with respect to the attribute being measured.

- Ask students to identify the potential positives and negatives when using a random sample to measure the attribute of an entire population. Examples of positives include the following:
  - the logistic ease of surveying, for example, 100 people instead of 70,000, and
  - the time commitment needed to survey a large population.

  Examples of negatives include the following:
  - the difficulty in obtaining a truly random sample, and
  - ensuring that the sample size is large enough to represent all relevant subpopulations of the entire population.

  For example, if the entire population is the 70,000 people attending a professional football game and the attribute of interest is the amount paid for a ticket, the sample size has to be large enough to include people seated in all areas of the stadium, people who purchased their tickets ahead of time and those that purchased tickets on the day of the game, season ticket holders, and so on.

- Ask students to describe situations in which the conditions of the survey of the sample population might not remain constant. For example, with the entire population of 70,000 people described above, measure their enjoyment of the game on a scale of 1 to 10 by taking a survey. Ask students to list possible reasons the attribute being measured could change given different situations. In the example situation, some reasons include the following.
    - the score of the game
    - the weather
    - the time of day
    - the increased (or decreased) interest in the game based on the amount of time remaining

  Students should conclude that if the goal is to compare how the score of the game affects people's enjoyment of the game, then it is important to measure people's enjoyment at times when the scores are different, but the other factors are as similar as possible. Or if the goal is to measure how the weather affects people's enjoyment of the game, then it is important to measure people's enjoyment at times when the weather is different, but the other factors are as similar as possible.

Key Academic Terms:

random sampling, population, sample, valid, draw inference, representative, expected value

Additional Resources:

- Activity: Mr. Brigg's class likes math
- Lesson: Sampling
- Lesson: Grade 7 mathematics module 5, topic C, lesson 13

10b

Data Analysis, Statistics, and Probability
Make inferences about a population using random sampling.
10. Examine a sample of a population to generalize information about the population. b. Compare sampling techniques to determine whether a sample is random and thus representative of a population, explaining that random sampling tends to produce representative samples and support valid inferences.

Guiding Questions with Connections to Mathematical Practices:

How can a sample best represent a population?

M.P.4. Model with mathematics. Construct a random sampling to best represent a population. For example, a sample of students in a school is random when all the names of the students are put in a bucket and selected randomly, rather than choosing one grade 7 math class as a sample. Additionally, identify the biases that may be introduced by different sampling methods.

- Ask students to develop methods of random selection and know that some methods might not be reasonable for certain populations. For example, when selecting names of students from a single class, writing the name of each student on a slip of paper and drawing at random to create a sample is reasonable. However, when creating a random sampling of a larger population, such as people with library cards for a public library, such a method is most likely logistically unreasonable. For such a population, library card numbers could be put into a list and a random number generator could be used to create a random selection of people based on their library card numbers.

- Ask students to identify biases that may be introduced by different sampling methods. For example, if the population being sampled is people with library cards and the method of selection is sorting the library card numbers in ascending order and then selecting the first 500 people from the list, a bias may be introduced since library card numbers may be assigned in order. Such a selection might result in selecting older users of the library, which may affect how representative the sample is, depending on the attribute being measured. Likewise, if the population of interest is users who upload videos to a website and the method of selection is to select the first 200 users who upload a video after 8:00 a.m., the data may be skewed based on the time zones of people who are active at that time of day.

Key Academic Terms:

random sampling, population, sample, valid, draw inference, representative, expected value

Additional Resources:

- Activity: Mr. Brigg's class likes math
- Lesson: Sampling
- Lesson: Grade 7 mathematics module 5, topic C, lesson 13

10c

Data Analysis, Statistics, and Probability
Make inferences about a population using random sampling.
10. Examine a sample of a population to generalize information about the population. c. Determine whether conclusions and generalizations can be made about a population based on a sample.

Guiding Questions with Connections to Mathematical Practices:

How does the size of a random sample affect the accuracy of a prediction or generalization about a population?

M.P.6 Attend to precision. Know that some populations may require a larger sample size than other populations when making a generalization or prediction. For example, a survey of 20 students is not likely to produce a good generalization about music preference for a school population of 1,200 students. However, surveying 20 orchestra members is likely to produce a good generalization for an orchestra of 50 total members. Additionally, identify when the size of a random sample is unnecessarily large for the size of a population.

- Provide students with a random sample size. Ask them to determine whether it is likely to be sufficient for making a prediction or generalization about different populations. For example, ask students to comment on whether a random sample of 12 would be sufficient for making a generalization about students in a classroom, members in a choir, residents in a city, and citizens of a county. In a classroom or a choir, it is likely that a random sample of 12 would be a large enough sample size to make a generalization. In a city or a country, a sample size of 12 would likely be too small to make a generalization about the population as a whole.

- Provide students with a variety of populations and ask them to comment on whether particular samples are likely to be too small, unnecessarily large, or appropriate. For example, ask students to comment on each of the following populations and sample sizes.
  - a survey of 50 players from a team of 60 about sport drink preferences
  - a survey of 3 seventh-grade students from a class of 30 about time spent on homework
  - a survey of 15 teachers from a staff of 40 about the number of pages of paper printed each week

  In the first instance, the sample is unnecessarily large because it surveys more than 80% of the entire population. By contrast, the sample size in the second instance is likely to be too small to accurately represent the entire population. The third instance represents a sample size that is likely to accurately represent the population without being unnecessarily large.

How can the conditions in which a random sample is collected affect the accuracy of a generalization or prediction about a population?

M.P.8 Look for and express regularity in repeated reasoning. Know that generalizations about a population may be affected based on the time of day, day of week, or month in which the sample is collected. For example, a survey about how much the residents of a neighborhood are charged for their monthly electricity bill may produce different generalizations or conclusions depending on whether the survey was conducted in January or June because of weather differences between those two months. Additionally, identify how conditions could be changed so that a random sample produces an accurate generalization or conclusion.

- Provide students with a situation that requires collecting a random sample. Ask them to explain how the time of day could affect the accuracy of the sample. For example, suppose that a restaurant owner wants to survey a random sample of customers about their satisfaction with the amount of time they have to wait. Students should be able to explain that satisfaction is likely to vary depending on whether the customers visited the restaurant during peak times (e.g., lunch or dinner) rather than non-peak times (e.g., mid-afternoon).

- Provide students with the details of how a random sample was collected. Ask them to suggest a change to the conditions so that a more accurate generalization could be made. For example, describe for students a situation in which a gym owner surveys 30 of the members in the gym at 7:00 A.M. to make a generalization about what time gym members wake up each morning. In this case, the results are likely to be inaccurate because of the relatively early time the survey is taken. The accuracy could be improved by surveying every tenth member that visits the gym throughout the day.

Key Academic Terms:

population, sample population, random sample

Additional Resources:

- Video: Random Sampling and Estimation: Lake Victoria
- Lesson: More about Sampling Variability

10d

Data Analysis, Statistics, and Probability
Make inferences about a population using random sampling.
10. Examine a sample of a population to generalize information about the population. d. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest, generating multiple samples to gauge variation and making predictions or conclusions about the population.

Guiding Questions with Connections to Mathematical Practices:

How can a random sampling help to draw inferences about a population?

M.P.7. Look for and make use of structure. Observe the data about a random sampling and apply the same likelihoods to the population to make predictions about an unknown characteristic of interest. For example, if multiple random samplings of fish in a lake have a mean length of 5 inches, scientists can infer that the mean length of all fish in the lake will be close to 5 inches. Additionally, know that the likelihoods and predictions made from data from a sample are estimates and their accuracy depends on the size of the sample as well as other factors.

- Ask students to use a random sample to make a prediction about a characteristic of a larger population. For example, students can ask their classmates about how many siblings each of them has. By calculating the mean, median, or other measures of center, students can make statistical inferences about the grade 7 student population as a whole. For example, if the mean number of siblings among their classmates is $1\frac{2}{3}$, students can infer that the mean number of siblings among the grade 7 student population as a whole is likely to be approximately $1\frac{2}{3}$.

- Ask students to discuss what factors pertaining to a random sample might increase or decrease the accuracy of predictions made about the population as a whole. Some factors that affect the accuracy of predictions include the following:
    - the size of the random sample compared to the size of the population
    - the conditions under which the characteristic is measured for the random sample
    - the degree to which the random sample truly reflects the population as a whole

How can simulations be used to see the variability of samples from a population?

M.P.4. Model with mathematics. Create a simulation to observe the variation of randomly selected samples from a population. For example, selecting a soccer player from a league where 40% of the players are 12 years old and 60% of the players are 13 years old can be simulated with a spinner that is 40% (or $\frac{4}{10}$) blue and 60% (or $\frac{6}{10}$) red. Each spin represents selecting either a 12-year-old (blue) or a 13-year-old (red). Record the number of blue spins and red spins and plot the results on a dot plot to analyze the variability. Additionally, identify the limitations of some simulations.

- Ask students to create a simulation to represent drawing a sample from a population and then use that simulation to represent possible outcomes. For example, a radio station randomly selects three callers to win a prize. Of the callers, $\frac{2}{3}$ are older than 40 and $\frac{1}{3}$ are younger than 40. A simulation can be created using a number cube where the numbers 1, 2, 3, and 4 represent the selection of callers older than 40 and the numbers 5 and 6 represent the selection of callers younger than 40. Ask students to simulate selecting three callers to be winners by rolling the number cube 3 times and recording how many of the winners are older than 40 and how many are younger than 40. They should repeat this simulation five times and record their winners with each simulation. Students can combine all their results to simulate the drawing of a sample approximately 100 times and see a distribution of the results.

- Ask students to discuss the limitations of simulations. In the previous example, it is likely that each caller can only win a single prize. After the first winner is selected, the fraction of remaining callers that are older than 40 and younger than 40 changes. After this change, the simulation is no longer entirely accurate because the probabilities associated with each result in the simulation do not change. Students should note that the accuracy of the simulation increases with a larger population.

Key Academic Terms:

random sampling, population, variability, inference, prediction, multiple, gauge

Additional Resources:

- Lesson: Sampling
- Activity: Gathering data to analyze
- Activity: Valentine marbles
- Lesson: Grade 7 mathematics module 5, topic C, lesson 14

10e

Data Analysis, Statistics, and Probability
Make inferences about a population using random sampling.
10. Examine a sample of a population to generalize information about the population. e. Informally explain situations in which statistical bias may exist.

Guiding Questions with Connections to Mathematical Practices:

What is a biased sample and how does it affect a generalization or prediction about a population?

M.P.2 Construct viable arguments and critique the reasoning of others. Know that a biased sample is a sample that does not accurately or fairly represent a population. For example, a sample population of 20 students in grade 7 is a biased sample if the intent is to make a generalization about students in an entire middle school with grades 6, 7, and 8. Additionally, observe that a biased sample can produce either an underestimate or an overestimate of a population or generalization.

- Ask students to determine whether a sample was biased by comparing information from the sample with information from the entire population. For example, provide students with the following table and ask them to explain whether the sample was biased.

	Number of Students	Number of Students Who Buy Hot Lunch
Sample	50	15
Population	266	160

In this case, it is likely that the sample was biased because only 30% of the students from the sample indicated that they buy hot lunch while about 60% of the entire population actually buys hot lunch.

- Provide students with two different sampling methods. Ask them to identify and explain which method is more likely to be biased. For example, suppose 30 students from a middle school with a total student population of 250 will be surveyed about whether they have a phone.

    - Method 1: survey all 30 students in 1 classroom during second period

    - Method 2: survey every eighth student that walks into the building in the morning

    In this case, Method 1 is likely to be biased because all 30 students in 1 classroom could be from the same grade. By contrast, Method 2 is more likely to fairly represent the student body.

How can “randomness” help ensure that a population sample is not biased?

M.P.4 Model with mathematics. Know that a random sample prevents statistical bias by allowing every member of the population to have the same chance of being selected. For example, making a generalization about sports preferences at a middle school is likely to be more accurate if seventy students are randomly selected to be surveyed than if seventy members of the football team are selected to be surveyed. Additionally, know that a random sample of a population may be obtained using more than one method.

- Provide students with a context that requires conducting a survey. Ask them to create at least two different ways to obtain a random sample. For example, suppose a mayor wants to conduct a random sample of the homeowners in her city. A variety of methods could be used. Some possibilities are listed.

    - surveying every homeowner with a house number that ends in 9

    - surveying every tenth homeowner from an alphabetical list of all homeowners

    - using computer software or a spreadsheet to randomly select 10 percent of all the homeowners

- Provide students with a variety of sampling methods. Ask them to identify which ones are unbiased. For example, the following list shows some sampling methods used to determine student plans to attend the musical.
  - randomly selecting 40 members of the school choir
  - randomly selecting 40 members of the school football team
  - randomly selecting 40 students exiting the building after school

  Students should be able to identify that only the third bullet point represents a sampling method that is unbiased. Although the first and second bullet points include an element of randomness, neither method will fairly represent the entire school population.

Key Academic Terms:

population, population sample, random sample, bias

Additional Resources:

- Lesson: Sampling in a fair way
- Article: What is statistical bias and why is it so important in data science?

11

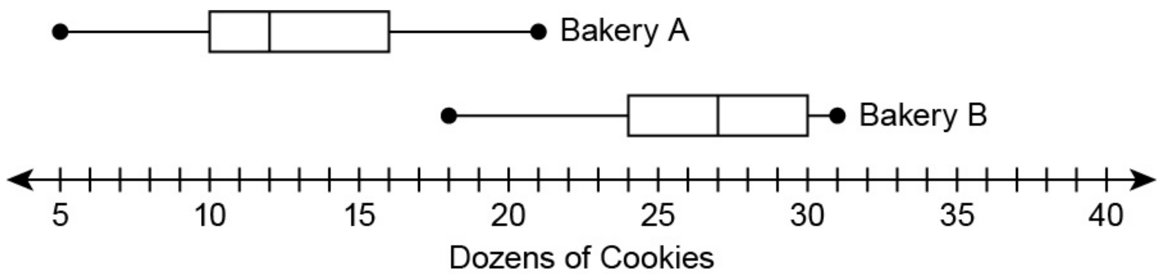
Data Analysis, Statistics, and Probability
Make inferences from an informal comparison of two populations.
11. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability.

Guiding Questions with Connections to Mathematical Practices:

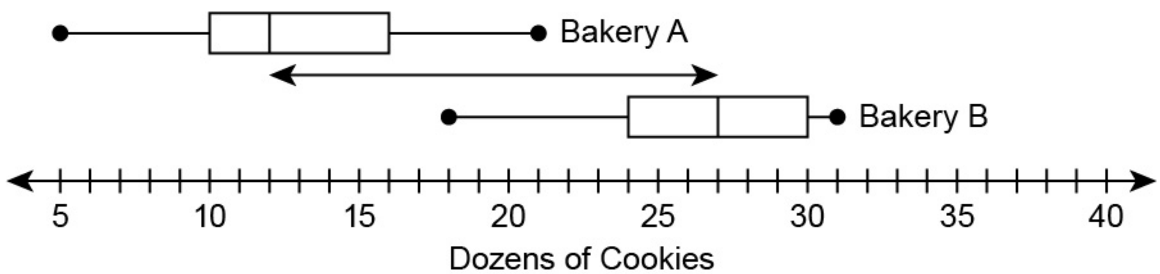
How can a measure of variability help to describe a numerical data distribution?

M.P.3. Construct viable arguments and critique the reasoning of others. Describe numerical distributions using measures of variability when all the measurements in the populations are known and no sampling is needed. For example, the median number of steps per day taken by a seventh-grade student is 500 steps less than the median number of steps per day taken by an eighth-grade student. Both students' data have the same interquartile range of 2,000 steps. Compare the data sets by using a measure of central tendency and a measure of variability by discussing the median and interquartile range. Additionally, note that expressing the distance between the medians as a multiple of the interquartile range can often give better comparisons.

- Ask students to compare two box plots that have the same interquartile range by estimating the distance between the medians expressed in terms of the interquartile range. For example, give students the box plots shown, which represent the numbers of dozens of cookies baked by two different bakeries.



Observe that plot A and plot B have the same interquartile range, 6 dozen cookies, and draw a horizontal line indicating the distance between the medians of the plots.



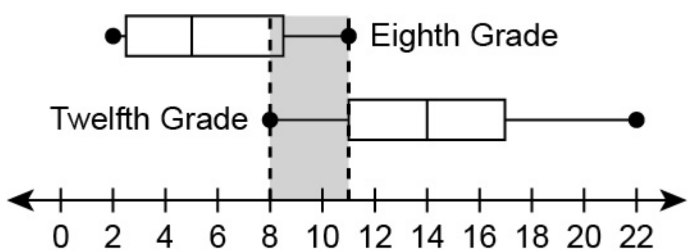
Ask students how many interquartile ranges would fit along the line. Have students cut out one of the boxes representing the interquartile range and use it to measure the length of the line. In this case, between 2 and 3 (about $2\frac{1}{2}$) interquartile ranges would fit along the line. Additionally, the length of the line (15 dozen cookies) can be divided by the interquartile range of 6 to get an estimate of $2\frac{1}{2}$.

- Ask students to calculate the median and interquartile range of two data sets and use those measures to compare the data sets. For example, the following two data sets show the number of text messages each of 12 eighth-grade students and 12 twelfth-grade students received last week.

Eighth-Grade Students
{2, 2, 2, 3, 3, 5, 5, 8, 8, 9, 10, 11}

Twelfth-Grade Students
{8, 8, 10, 12, 12, 14, 14, 16, 16, 18, 19, 22}

The first quartile for eighth-grade students is 2.5, and the third quartile is 8.5, resulting in an interval of 6. For twelfth-grade students, the first quartile is 11, and the third quartile is 17, which also results in an interval of 6. Therefore, the interquartile range of each of the two data sets is 6. An IQR of 6 means that the middle 50% of the data lie in an interval of length 6 for each data set. The median of the data for the eighth-grade students is 5, while the median of the data for the twelfth-grade students is 14. The difference between the two medians is 1.5 times the interquartile range, and there is very little visual overlap between the box plots of the two data sets.

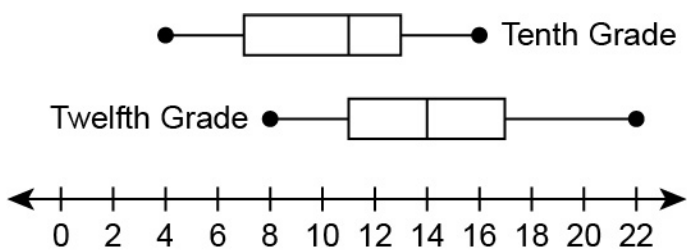


From the diagram, the shading shows that approximately the upper 25% of the eighth-grade students' data and the lower 25% of the twelfth-grade students' data overlap.

In contrast, the following data set shows the number of text messages each of 12 tenth-grade students received last week.

Tenth-Grade Students
{4, 5, 5, 9, 9, 10, 12, 12, 13, 13, 15, 16}

The interquartile range for this data set is also 6, and the median is 11. Comparing the tenth-grade students to the twelfth-grade students, the gap between the two medians is 0.5 times the interquartile range. This suggests that the data sets are closer together and will have much more overlap. This can be seen in the box plot shown.



- Ask students to predict the degree of overlap based on the means and the interquartile ranges. For example, consider the following questions:
    - The median of one data set is 45 and the median of another data set is 55. How much do the data sets overlap?

      Solution: It is impossible to tell based on only the medians. The medians are 10 units apart, but that does not indicate anything about the rest of the data values.
    - When told that the interquartile range of both data sets is 3, what does that say about the degree of overlap?

      Solution: The distance between the medians is more than three times the interquartile range. This suggests that there is probably very little overlap in the two data sets (though it does not guarantee this).
    - When told that the interquartile range of both sets of data is 20, what does that say about the degree of overlap?

      Solution: The distance between the medians is half the interquartile range. This suggests that there is most likely a large amount of overlap between the two data sets (though it is not guaranteed in this case either).

  - Which pair of data sets has more overlap: two data sets that have medians of 50 and 200 and have a common interquartile range of 100, or two data sets that have medians of 5 and 10 and have a common interquartile range of 1?

    Solution: The first data set has two centers that are extremely far away from each other. However, this distance of 100 is only 1.5 times the interquartile range. The second data set has two centers that are only 5 units away, but this distance is 5 times the interquartile range.

  Therefore, it is most likely that the first data set has a greater degree of overlap. Expressing the distance between the medians as a multiple of the interquartile range helps to rescale the size of the numbers so that we can compare data sets that have very different numbers.

Key Academic Terms:

data distribution, variability, range, interquartile range, measure of central tendency, deviation, measure of variability

Additional Resources:

- Activity: College athletes
- Activity: Offensive linemen
- Lesson: Grade 7 mathematics module 5, topic D, overview
- Lesson: Comparing distributions
- Lesson: Grade 7 mathematics module 5, topic D, lesson 22
- Lesson: Comparing populations using samples
- Lesson: Comparing sets of data

12

Data Analysis, Statistics, and Probability
Make inferences from an informal comparison of two populations.
12. Make informal comparative inferences about two populations using measures of center and variability and/or mean absolute deviation in context.

Guiding Questions with Connections to Mathematical Practices:

How can numerical data distributions of random samples be compared using measures of center?

M.P.3. Construct viable arguments and critique the reasoning of others. Analyze the distributions of the random samples of data and compare the medians or means. For example, the mean rainfall amount in a random sample of cities in Alabama in the month of April is less than the mean rainfall amount in a random sample of cities in Alabama in the month of March, so March generally gets more rainfall than April in Alabama. Additionally, demonstrate that using different measures of center for comparison can appear to change the relationship between the two data distributions.

- Ask students to compare the medians and means of two given data sets. For example, the following data sets show the number of games of chess each of 8 randomly selected members of chess clubs played in 2 different chess clubs last week.

Games Played by Members in the Monday club
{4, 4, 5, 6, 6, 9, 11, 13}

Games Played by Members in the Wednesday club
{5, 8, 10, 11, 13, 14, 14, 15}

Ask students to find the mean and median of the number of games played by members in each chess club. For the Monday club, the mean is 7.25 and the median is 6. For the Wednesday club, the mean is 11.25 and the median is 12. Then, ask students to write comparisons of the number of games played by the entire population of each chess club. Example comparisons might include the following:

  - Because the mean number of games played by the sample from the Monday club is less than the mean played by the sample from the Wednesday club, it is likely that members of the Monday club play fewer games of chess weekly than members of the Wednesday club.

  - Because the median number of games played by the sample from the Wednesday club is twice the median played by the sample from the Monday club, it is likely that members of the Wednesday club play twice as many games weekly as members of the Monday club.

- Ask students to discuss how using different measures of center for comparison can appear to change the relationship between data distributions. In the previous example, while both the mean and median of the Wednesday club's data are greater than the respective measures of the Monday club's data, the magnitude of the differences between the respective measures has different implications. The following data sets show the number of permits purchased each day for 7 days for 2 state parks.

  Permits for Park A
  {2, 14, 15, 18, 20, 20, 23}

  Permits for Park B
  {3, 3, 5, 19, 20, 20, 21}

  Here, the mean for Park A is 16 and the median is 18, while the mean for Park B is 13 and the median is 19. The conclusions that can be drawn about the number of permits that will be purchased over the course of an entire year greatly depend on which measure of central tendency is used. For example, if the mean values are used, then a person would likely conclude that Park A will see a significantly greater number of permits purchased over the course of a year than Park B. If the medians are used to draw a conclusion, then a person would likely conclude that Park B will see a slightly greater number of permits purchased over the course of a year than Park A.

What is the mean absolute deviation of a data set and how does it differ from the mean?

M.P.4. Model with mathematics. Know that while the mean is a measure of center, the mean absolute deviation is a measure of variation that indicates the mean amount that the data differ from the mean of the data. For example, the data set {4, 4, 6, 8, 8, 12, 14} represents the ages in years of children at a restaurant on one occasion. The mean age of the children is 8 years, while the mean absolute deviation of the ages is approximately 2.86, which indicates that, on average, each child's age is within 2.86 years of 8. Additionally, know that the mean absolute deviation depends solely on the distance each value is from the mean and remains the same for data sets with the same distributions. For example, the mean absolute deviations of {1, 3, 4, 8} and {11, 13, 14, 18} are equal.

- Ask students to examine the term “mean absolute deviation” word by word.
    - The word “deviation” means “to differ from.”
    - The word “absolute” refers to absolute value, which is a reminder that mean absolute deviation depends on the distance between two values and that distance is always positive.
    - The word “mean” refers to the value from which the distance is calculated.

  Students should observe that the term “mean absolute deviation” functions almost as a set of instructions for how to find it.

- Ask students to calculate the mean absolute deviation of a given data set. The following data set shows the number of sandwiches sold by a sandwich shop each day from noon to 1:00 p.m. for 10 consecutive days.

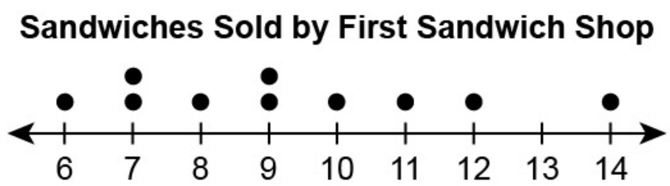
  $$\{6, 7, 7, 8, 9, 9, 10, 11, 12, 14\}$$

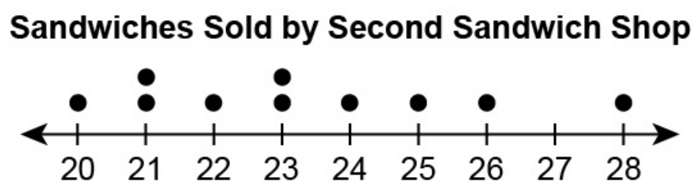
  Have students first calculate the mean. For the data set shown, the mean is 9.3. Then, have students calculate the distance from each value in the set to the mean. For example, the first value in the data set, 6, is $9.3 - 6 = 3.3$ units from the mean. The distance from the mean for each value in the set is shown.

  $$\{3.3, 2.3, 2.3, 1.3, 0.3, 0.3, 0.7, 1.7, 2.7, 4.7\}$$

  Have students find the mean of the 10 distances, which is the mean absolute deviation. In this case, the mean absolute deviation is 1.96.

- Ask students to calculate the mean absolute deviation of two given data sets presented as dot plots. For example, the first dot plot shown represents the data set about sandwiches given previously. The second dot plot represents the same information but for a different sandwich shop.





While the mean is different for each data set (the mean for the second data set is 23.3), the mean absolute deviation is the same for each data set because they have the same distribution. Students should know that it is the distance from the mean, not the actual values in the set nor the actual value of the mean, that is relevant for determining the mean absolute deviation.

- Create a number line on the floor using masking tape and ask students to model data sets by standing at the points in the data set. For example, illustrate the data set {1, 2, 5, 9, 13} by asking a student to stand at each of those points on the number line. Ask each student to count how many steps away they are from the mean value of 6. Record each student's response, then calculate the mean of the responses. This calculates the mean absolute deviation to be 4. Then, ask students to model another data set that has the same mean, but a different mean absolute deviation. For example, illustrate the data set {4, 5, 6, 7, 8}. Ask each student to count how many steps away they are from the mean value of 6, record each response, and then calculate the mean of the responses to get a mean absolute deviation of 1.2. Ask students to describe how crowded they were when modeling each data set and how that relates to the mean absolute deviation.

How can numerical data distributions of random samples be compared using measures of variability?

M.P.3. Construct viable arguments and critique the reasoning of others. Analyze the distributions of data and compare the mean absolute deviation, range, or interquartile range. For example, the average monthly precipitation in a certain year in Montgomery is between 2.58 inches per month and 6.39 inches per month, and the average monthly precipitation in that same year in Birmingham is between 3.23 inches per month and 6.10 inches per month. The variability of monthly precipitation is greater in Montgomery, with a range of 6.39 in. – 2.58 in. = 3.81 in., than in Birmingham, with a range of 6.10 in. – 3.23 in. = 2.87 in., because Montgomery has the greater range. Additionally, demonstrate that using different measures of variability for comparison can appear to change the relationship between the two data distributions.

- Ask students to compare measures of variability of two given data sets. For example, the following data sets show the number of absences in each of a teacher's 6 classes on both Monday and Tuesday.

  Absences on Monday
  $\{1, 2, 2, 3, 4, 6\}$

  Absences on Tuesday
  $\{0, 0, 1, 1, 2, 2\}$

  Ask students to find the range, interquartile range, and mean absolute deviation of the absences on each day. For Monday, the range is 5, the interquartile range is 2, and the mean absolute deviation is $1\frac{1}{3}$. For Tuesday, the range is 2, the interquartile range is 2, and the mean absolute deviation is $\frac{2}{3}$. Ask students to write comparisons of the number of absences among all the teachers' classes on Monday and Tuesday. Example comparisons might include the following:

  - The interquartile range of the number of absences in all teachers' classes on that Monday and Tuesday should be approximately equal.
  - Teachers should expect a much greater range of absences for Monday than for Tuesday.

- Ask students to discuss how using different measures of variability for comparison can appear to change the relationship between data distributions. In the previous example, when comparing the data distributions using the range, the data distributions appear quite different. The range of the number of absences on Monday is more than twice the range of the number of absences on Tuesday. Likewise, the distribution of absences on Monday appears more variable than on Tuesday based on the mean absolute deviation. However, when comparing the data distributions using the interquartile range, the data distributions appear to have the same variability. The interquartile ranges are both equal to 2.

Key Academic Terms:

random sampling, measures of center, measures of variability, mean absolute deviation

Additional Resources:

- Activity: College athletes
- Activity: Offensive linemen
- Activity: Does the percentage of people who walk to work in cities vary with population size?
- Activity: Exploring sampling variability—Higher education attainment across the United States
- Lesson: Comparing populations—What are center, shape and spread?
- Lesson: Measures of center and variability fluency practice
- Lesson: Comparing distributions
- Video: Mean absolute deviation

13

Data Analysis, Statistics, and Probability
Investigate probability models.
13. Use a number from 0 to 1 to represent the probability of a chance event occurring, explaining that larger numbers indicate greater likelihood of the event occurring, while a number near zero indicates an unlikely event.

Guiding Questions with Connections to Mathematical Practices:

How can the probability of a chance event represent the likelihood of that event occurring?

M.P.7. Look for and make use of structure. Connect the probability of a chance event to a number between 0 and 1, a percentage, or a ratio and observe that the numbers between them represent likelihood on a scale from not at all likely to very likely, with a probability of 0.5 meaning the event is neither likely nor unlikely. For example, a probability of 100% means an event absolutely will happen, and a probability of 99% means it is very likely. On the other hand, a probability of 0% means an event absolutely will not happen, and a probability of 1% means the event is very unlikely to happen. Additionally, observe that any value between 0 and 1 could be a probability, not just benchmark fractions such as $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

- Ask students to list events that have probabilities that could be described as impossible, unlikely, neither likely nor unlikely, likely, and certain. For each of the five categories, ask students to list one or two events. Possible events, from impossible to certain, could include the following:

    - drawing a random number that is not divisible by 1
    - people living on Mars in the next 10 years
    - a rolled number cube landing on an even number
    - a randomly selected day from next week being a school day
    - a randomly selected U.S. coin having a value of at least 1¢

  Students should then read their events, in random order, to a classmate and have the classmate order the events from least likely (impossible) to most likely (certain). Additionally, note that an example of a likely or certain event can be transformed into an example of an unlikely or impossible event by focusing on the opposite of the event. For example, a randomly selected day from next week being a school day is a likely event. A randomly selected day from next week NOT being a school day is an unlikely event.

- Ask students to describe how to change a particular given event that has a probability of 0.5 so the new probability is different than 0.5. For example, given the event of a number cube that is rolled landing on an even number, ask students to describe a change to the event that would make it an event with a probability less than 0.5. A possible change for the given event might be the number cube landing on the number 3. Additionally, given the event of randomly selecting a red marble from a bag with 10 red, 10 green, and 10 blue marbles being red, ask students to describe a change to the event or to the experiment that would make it an event with a probability greater than 0.5. A possible change for the given event might be the selected marble being either red or blue. A possible change for the experiment might be increasing the number of red marbles in the bag.

Key Academic Terms:

probability, likelihood, chance event, theoretical probability, experimental probability, ratio, percent, fraction, decimal

Additional Resources:

- Lesson: Grade 7 mathematics module 5, topic A, lesson 1
- Video: Probability: Tell the future

14a

Data Analysis, Statistics, and Probability
Investigate probability models.
14. Define and develop a probability model, including models that may or may not be uniform, where uniform models assign equal probability to all outcomes and non-uniform models involve events that are not equally likely. a. Collect and use data to predict probabilities of events.

Guiding Questions with Connections to Mathematical Practices:

How can a probability model be used to help determine the probabilities of events?

M.P.4. Model with mathematics. Construct a probability model to match the likelihoods of the outcomes of an event and perform the experiment to determine the probabilities of each outcome. For example, to represent the event that a drummer is randomly selected from a school band, first the number of students in the band is needed, as well as the number of drummers. Then, a probability model can be simulated by using a slip of paper for each student in the band, writing the word "drummer" on the appropriate number of slips of paper, and drawing the slips of paper from a bag. Additionally, simulate a uniform probability model in a variety of ways.

- Ask students to construct a uniform probability model for a given experiment. An example event might be that a randomly selected state is Alabama or a state adjacent to Alabama. There are 50 states, and each one is equally likely to be selected. The probability model can be shown with the following table.

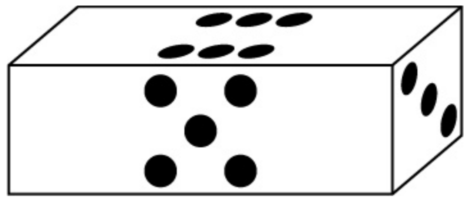
State	Alabama	Alaska	Arizona	…	Wyoming
Probability	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$	…	$\frac{1}{50}$

- Ask students to determine the probability of an event using a uniform probability model. Using the example above of the 50 states, 5 of them are either Alabama or a state neighboring Alabama (Tennessee, Georgia, Florida, or Mississippi). Therefore, the probability that a randomly selected state is Alabama or a state adjacent to Alabama is $\frac{5}{50}$, or $\frac{1}{10}$. This can be simulated by putting 9 red discs and 1 blue disc in a bag and selecting 1 at random. Students can conduct the experiment repeatedly and record the results after each disc selection. Students can discuss other possible numbers of outcomes that could be used in the model. For example, a model with 12 outcomes would not be able to accurately represent the event, because 12 cannot be divided into 10 equal, whole parts. However, a model with 100 outcomes would be able to represent the event since 100 is a multiple of 10.

How can a probability model be developed for a chance process that does not have equally likely probabilities of outcomes?

M.P.4. Model with mathematics. Construct a probability model to match the likelihoods of the outcomes of a chance process and observe the frequencies of each outcome to find the approximate probabilities. For example, try spinning a spinner that is divided into different-sized sections or flipping an object that does not have equally likely chances of landing on each side, e.g., a marshmallow or an eraser. Additionally, use multiple probability models in succession to approximate probabilities of an event occurring.

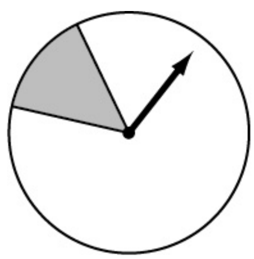
- Ask students to create a probability model based on the results of a chance process that does not have equally likely probabilities of outcomes. For example, give students a rectangular prism with a length of 6 inches and a width and height of 2 inches and faces labeled with dots numbering 1 to 6. Ask students to "roll" it and ask them to record which face is on top.



Ask students to use their results to create a probability model and to show the model using a table. A template for such a table is shown.

Top Face	1	2	3	4	5	6
Probability						

- Ask students to construct a probability model using results from conducting the actual process. For example, use the experiment of spinning the spinner shown.



Using the above example, a student might conduct 30 spins and find that 5 of them result in the shaded region. Ask students to show this by constructing a probability model like the table shown.

Region	Shaded	Unshaded
Probability	$\frac{5}{30}$	$\frac{25}{30}$

- Ask students to compare two probability models and determine their validity by conducting an experiment. For example, ask students to flip 3 coins. It is natural for students to come up with the following two uniform probability models.

Number of Heads	0	1	2	3
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Result	HHH	HHT	HTH	HTT	TTT	TTH	THT	THH
Probability	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

There are 3 out of 8 results in the second table that have two heads. Therefore, there is a discrepancy between the tables and at least one of them must be incorrect. Ask the class to determine which one is more likely to be incorrect by performing the experiment 80 times and recording the results. Note that it is possible to have an experiment where not all outcomes have uniform probability.

Key Academic Terms:

probability model, frequencies, uniform, discrepancy, outcome, ratio, percent, fraction, decimal

Additional Resources:

- Lesson: Compound events—Visual displays of sample spaces
- Activity: How many buttons?
- Activity: Tossing cylinders
- Video: Probability: remove one

14b

Data Analysis, Statistics, and Probability
Investigate probability models.
14. Define and develop a probability model, including models that may or may not be uniform, where uniform models assign equal probability to all outcomes and non-uniform models involve events that are not equally likely. b. Compare probabilities from a model to observed frequencies, explaining possible sources of discrepancy.

Guiding Questions with Connections to Mathematical Practices:

How can observed frequencies and probabilities from a model be compared?

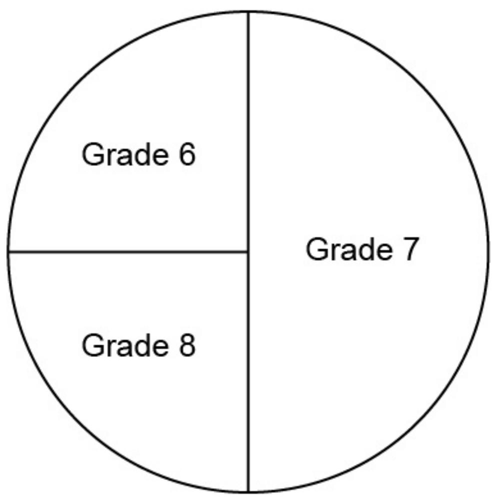
M.P.3. Construct viable arguments and critique the reasoning of others. Use ratios and proportional reasoning to compare the frequencies and probabilities of an event and note any discrepancies, explaining their possible cause. For example, an event with a theoretical probability of 0.25 is represented by spinning a spinner that has four equal-sized sections 100 times and recording the section it lands on each time. An experimental probability of 0.18 shows a discrepancy that could have been caused by chance or by a faulty or unfair spinner, as $0.18 \neq 0.25$. Additionally, know that the amount of the discrepancy affects the likelihood that the discrepancy is caused by chance versus an external factor.

- Ask students to conduct an experiment for which they can calculate the theoretical probability. For example, students can conduct an experiment in which a number cube is rolled and the result recorded. The uniform probability model can be shown with the following table.

Result	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

There are two results that are greater than or equal to 5. Therefore, the theoretical probability that a number greater than or equal to 5 is rolled is $\frac{2}{6}$, or $\frac{1}{3}$. Ask students to calculate, based on the theoretical probability, how many times they would expect a number greater than or equal to 5 to be rolled in 30 trials. In this case, the expected number is $\frac{1}{3} \times 30 = 10$. Then, ask students to conduct the experiment 30 times and record the number of times a number greater than or equal to 5 is rolled. Students can compare this number to the expected number.

- Ask students to create an experiment with a non-uniform probability model. For example, ask students to create a spinner with different-sized sections to represent the chance of randomly selecting a grade 7 student out of a group of students that is 25% grade 6, 50% grade 7, and 25% grade 8. An example spinner is shown.



Note that the percentage for each grade matches the size of each section on the spinner. Students spin 50 times with the results of 12 spins landing on grade 6, 24 spins landing on grade 7, and 14 spins landing on grade 8. To compare the observed frequencies to the expected 50% for grade 7, find the percentage that represents the 24 spins out of 50 spins. The observed frequency is 48%, which is very near the expected 50%.

- Ask students to discuss possible reasons for discrepancies between the expected and actual number of times an event occurs. Possible reasons include the number of trials conducted, unfairness in the experiment design, and chance. Possible reasons for the discrepancy should relate to the size of the discrepancy. For example, if the experiment described above is conducted 300 times and the expected value is 150, and the actual value is 146, the discrepancy can most likely be explained by chance; the actual value is extremely close to the expected value.

Key Academic Terms:

probability model, frequencies, uniform, discrepancy, outcome, ratio, percent, fraction, decimal

Additional Resources:

- Lesson: Write-It Wednesday—Probability
- Activity: Rolling dice
- Lesson: Determine probabilities

15a

Data Analysis, Statistics, and Probability
Investigate probability models.
15. Approximate the probability of an event using data generated by a simulation (experimental probability) and compare it to the theoretical probability. a. Observe the relative frequency of an event over the long run, using simulation or technology, and use those results to predict approximate relative frequency.

Guiding Questions with Connections to Mathematical Practices:

What is the difference between theoretical probability and experimental probability?

M.P.2. Reason abstractly and quantitatively. Compare experimental and theoretical probabilities. For example, a fair coin has a theoretical probability of landing on heads one-half, or 50%, of the time because it is one of only two possible outcomes. If actually conducting the experiment of flipping the coin 100 times results in the coin landing on heads 42 times, then the experimental probability is 42%. Additionally, know that the more times an experiment is conducted, the closer the experimental probability should be to the theoretical probability.

- Ask students to calculate theoretical probabilities associated with an event and then conduct the experiment and record the experimental probabilities. For example, using a 6-sided number cube, students can calculate the theoretical probability of rolling a number less than 3 to be $\frac{2}{6}$ or $\frac{1}{3}$. Then, students can conduct the experiment by rolling the number cube 10 times and recording the number of times the roll is a number less than 3. Ask students to share whether their experimental probability was less than, equal to, or greater than the theoretical probability.

- Ask students to calculate the difference between the experimental and theoretical probabilities of an event as the number of times the experiment was conducted increases. For instance, using the above example, record the experimental results from a single student with 10 trials and calculate the difference between the experimental probability and the theoretical probability. Then, include the results from another student's 10 trials and re-calculate the experimental probability and compare it to the theoretical probability. The table shown could represent this information after 3 students have reported the results of the 30 trials.

Number Cube Probabilities

Total Trials	Total Successful Events	Experimental Probability	Theoretical Probability	Difference between Experimental and Theoretical Probabilities
10	2	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{2}{15}$
20	5	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$
30	9	$\frac{3}{10}$	$\frac{1}{3}$	$\frac{1}{30}$

Ask students to discuss the trend of the difference between the two probabilities and whether it can be guaranteed that the two probabilities will ever be exactly the same.

How can relative frequencies help to approximate the probability of a chance event?

M.P.7. Look for and make use of structure. Collect data on a chance process to find the long-run relative frequency to compute the probability of the event. For example, randomly selecting one marble at a time from a bag full of marbles, recording the color, and replacing it in the bag can help to approximate the probability of red marbles being drawn from the bag by dividing the frequency of red by the total number of times selections were drawn. Additionally, demonstrate that the difference between the relative frequency and the theoretical probability tends to decrease as the total number of repetitions of the experiment increases.

- Ask students to estimate the probability of an event for which the theoretical probability is not calculable. For example, give each student or group of students a bag containing a total of 10 discs, some of which are yellow and the rest of which are blue. Students cannot know the contents of the bag other than that it contains 10 discs. Have students randomly select a disc, record its color, and return it to the bag. After each selection, have students use their results to estimate the proportion of yellow discs in the bag. For example, consider the first 10 selections listed below:

  {yellow, yellow, blue, yellow, yellow, yellow, yellow, blue, blue, yellow}

  After the first selection, the students might organize the results in the following table.

Draw Number	Color Drawn	Total Yellow So Far	Total Blue So Far	Current Experimental Probability for Yellow
1	yellow	1	0	1

  After a few more selections, the table might look like the following.

Draw Number	Color Drawn	Total Yellow So Far	Total Blue So Far	Current Experimental Probability for Yellow
1	yellow	1	0	1
2	yellow	2	0	1
3	blue	2	1	$\frac{2}{3}$
4	yellow	3	1	$\frac{3}{4}$

  The experimental probabilities of drawing a yellow disk, calculated after each of the first 10 draws, are shown.

$$\{1, 1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{3}{4}, \frac{2}{3}, \frac{7}{10}\}$$

  After each of the first 2 selections, it would be reasonable to predict that all the discs in the bag are yellow since that was the only color drawn. After the third selection, it would be reasonable to predict that about $\frac{2}{3}$ of the discs in the bag are yellow since 2 of the first 3 discs selected were yellow.

- Ask students to discuss how the accuracy of the predictions changes as the number of times the event is repeated increases. Using the data shown above as an example, it would be reasonable to conclude that 7 of the 10 discs are yellow based on the results of the first 10 selections. However, a prediction made after either of the first 2 selections would likely be that 10 of the 10 discs are yellow, while a prediction made after the first 5 selections would likely be that 8 of the 10 discs are yellow. Students should note that the predictions change as the number of trials increases. Show students the number of discs of each color in the bag and ask them to calculate the theoretical probability and then compare the predictions to the theoretical probability. Students should note the accuracy of their predictions tends to increase (but not always) as the number of trials increases.

How can the probability help to predict the approximate frequency?

M.P.7. Look for and make use of structure. Connect probability to relative frequency by observing that the frequency of an event is divided by the total number of events to find probability. For example, an event with a probability of 25% means that 1 out of 4 selections should match that event, and the frequency can be approximated using ratios and proportional thinking. Additionally, demonstrate that the relative frequency and the probability converge as the total number of events increases.

- Ask students to calculate the probability of an event and interpret that probability in terms of expected frequency. For example, given a number cube, the probability that a single roll is a 5 or 6 is $\frac{2}{6}$ or $\frac{1}{3}$. Ask students to use the expected probability to estimate the number of times the event of interest will occur in a given number of trials. For example, given 3 rolls, the student should expect a roll of 5 or 6 one time, since $\frac{1}{3} \times 3 = 1$. Given 60 rolls, the student should expect a roll of 5 or 6 twenty times, since $\frac{1}{3} \times 60 = 20$.

- Ask students to calculate and use the probability of an event to predict the frequency of the event given 5, 10, 25, and 50 trials. Then, ask students to conduct 50 trials and record the actual frequency of the event of interest after 5, 10, 25, and 50 trials. Ask students to compare the prediction based on the probability with the actual frequency and make observations about the accuracy of the predictions as the number of trials increases.

Key Academic Terms:

probability, frequency, data, relative frequency, approximate, ratio, percent, fraction, decimal

Additional Resources:

- Activity: How many buttons?
- Activity: Tossing cylinders

16a

Data Analysis, Statistics, and Probability
Investigate probability models.
16. Find probabilities of simple and compound events through experimentation or simulation and by analyzing the sample space, representing the probabilities as percents, decimals, or fractions. a. Represent sample spaces for compound events using methods such as organized lists, tables, and tree diagrams, and determine the probability of an event by finding the fraction of outcomes in the sample space for which the compound event occurred.

Guiding Questions with Connections to Mathematical Practices:

How can the sample space of a simple event be represented?

M.P.1. Make sense of problems and persevere in solving them. List all possible outcomes of a simple event in an organized list. For example, the possible outcomes for a coin flip are heads or tails, and the sample space is {heads, tails}. Additionally, know the difference between the outcomes of a particular event and the outcomes of the entire sample space.

- Ask students to identify the sample space of an experiment. For example, give students any of the following experiments:
  - A number cube is rolled.
  - A marble is randomly drawn from a bag containing 1 red, 1 blue, and 1 green marble and its color is noted.
  - A player is randomly assigned to play one of the nine positions in baseball.

  The sample spaces in each case would be {1, 2, 3, 4, 5, 6}, {red, blue, green}, and {first base, second base, third base, catcher, pitcher, shortstop, left field, center field, right field} respectively.

- Ask students to identify the outcomes of a sample space that are included in a simple event. For example, give students the experiment of picking a random letter of alphabet and the event of choosing a letter that can be used to complete the word __ARE. The sample space consists of all 26 letters of the alphabet. The event consists of the outcomes {B, C, D, F, H, M, P, R, T, W, Y}.

How do compound events relate to simple events?

M.P.7. Look for and make use of structure. Know that a compound event is made up of multiple simple events. For example, rolling a 5 on a number cube once is a simple event, and rolling a 5 on a number cube twice is a compound event. Another example of a compound event is to first draw a face card out of a deck and then choose a red marble out of a bag. Additionally, know that some experiments allow for repeated selections and some do not.

- Ask students to identify simple events and compound events. Help students to know that not all compound events will be explicitly stated as a combination of simple events. For example, randomly choosing a grade 7 student out of all the students at a school is a simple event, whereas choosing a random student who is both a grade 7 student and plays an instrument is a compound event because it is made up of two simple events. Another example is rolling a sum of 2 on a pair of number cubes. Although it is not stated as a combination of simple events, rolling a sum of 2 requires the two simple events of rolling a 1 on the first number cube and rolling a 1 on the second number cube thereby making it a compound event.

- Ask students to create compound events of a simple experiment based on simple events and identify the outcomes that make up that compound event. For example, give students a spinner that is numbered from 1 to 9 and give them the following simple events:

    - Event A: The spinner lands on an odd number.
    - Event B: The spinner lands on a prime number.
    - Event C: The spinner lands on a number greater than 2.

  One compound event that students can create is the compound event made up of Event A and Event C. This event includes all odd numbers that are greater than 2 and consists of the set of outcomes, {3, 5, 7, 9}. Further, note that this event has a probability of $\frac{4}{9}$ because it consists of 4 out of 9 equally likely outcomes.

- Ask students to identify compound events in a compound experiment. For example, Albert has purple, orange, green, and black crayons that he can use to color a picture of a house. He randomly chooses one color for the roof and a different color for the walls. The event that Albert chooses to color the roof green and the walls purple is a compound event. It combines the event of randomly choosing green for the roof and the event of randomly choosing purple for the walls. Further, observe that the compound event of choosing green for the roof and green for the walls is not allowed in this case because the experiment requires that Albert chooses two different colors. Contrast this with the experiment of rolling a number cube two times, where it is possible to roll the same number twice.

How can the sample space of a compound event be represented?

M.P.1. Make sense of problems and persevere in solving them. Construct an organized list, a table, or a tree diagram to represent the sample space of a compound event. For example, a tree diagram can be used to show all possible sandwich combinations when there are 2 choices of bread, 3 choices of meat, and 4 choices of cheese. Additionally, observe that different representations have different advantages and disadvantages.

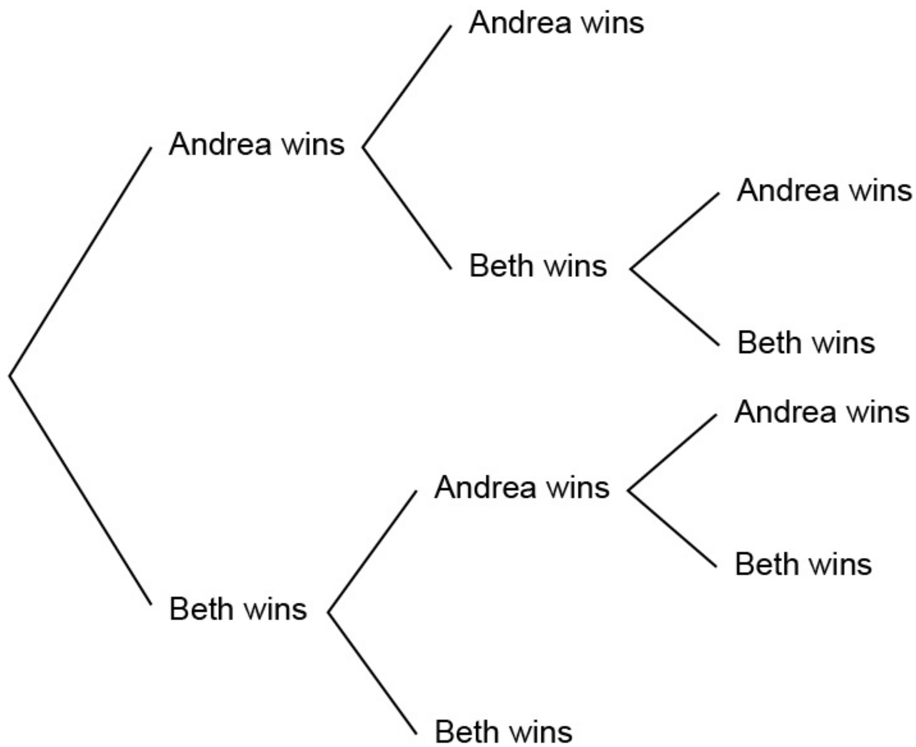
- Ask students to construct an organized list to find the sample space of a compound event. For example, find the sample space when choosing a first book to read and a second book to read from a set of five books. Label the books as A, B, C, D, and E. Encourage students to find an organized manner in which to list the selected books to be certain that all combinations have been selected. One way of doing this is to start by listing all combinations in which A was the first book selected, {AB, AC, AD, AE}. Then, list all combinations in which B was the first book selected, then C, and so on. The full sample space is {AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED}.

- Ask students to construct a table to represent the sample space of a compound event. For example, Asher owns three pairs of pants and four shirts. The table shown shows the sample space of different outfits Asher can choose.

	Red Shirt	**White Shirt**	**Pink Shirt**	**Orange Shirt**
Blue Pants	Blue Pants and Red Shirt	Blue Pants and White Shirt	Blue Pants and Pink Shirt	Blue Pants and Orange Shirt
Black Pants	Black Pants and Red Shirt	Black Pants and White Shirt	Black Pants and Pink Shirt	Black Pants and Orange Shirt
Brown Pants	Brown Pants and Red Shirt	Brown Pants and White Shirt	Brown Pants and Pink Shirt	Brown Pants and Orange Shirt

Observe that the table can easily be extended to include more pants and shirt options. However, adding a third type of clothing, such as shoes, to choose would make using a table much more difficult. In general, a table is very good for handling two layers of decisions even if each layer is very large.

- Ask students to construct a tree diagram to create the sample space of a compound event. For example, a tennis tournament uses a best-of-3 format for the finals. Andrea and Beth play several games against each other until one of them wins two games. A tree diagram for this situation is shown.



Note that tree diagrams have the advantage of handling several layers of decisions, including situations where not all layers are identical. Also note that tree diagrams have the disadvantage of being very cumbersome and taking up a lot of space.

Key Academic Terms:

compound event, simple event, sample space, simulation, ratio, tree diagram, percent, fraction, decimal

Additional Resources:

- Tutorial: Probability of compound events
- Activity: Tetrahedral dice
- Lesson: Compound probability
- Lesson: Grade 7 mathematics module 5, topic A, lesson 7

16b

Data Analysis, Statistics, and Probability
Investigate probability models.
16. Find probabilities of simple and compound events through experimentation or simulation and by analyzing the sample space, representing the probabilities as percents, decimals, or fractions. b. Design and use a simulation to generate frequencies for compound events.

Guiding Questions with Connections to Mathematical Practices:

How can simulations help to estimate probabilities for compound events?

M.P.4. Model with mathematics. Design a model and use a simulation to collect frequency data about a compound event. For example, suppose the probability of an event happening is 0.6. To simulate the probability of the event occurring two times in a row, use a spinner with one section that is $\frac{6}{10}$ of the circle and one section that is $\frac{4}{10}$ of the circle. Use the spinner to simulate the event happening two times in a row and record the frequency data. Additionally, observe that devices that do not naturally have the correct number of outcomes for a situation can still be used by either ignoring certain outcomes to reduce the number of outcomes or by repeating the trial to increase the number of outcomes.

- Ask students to describe how various probability devices could be used to simulate a simple event. For example, a weather forecaster correctly predicts the weather with a probability of $\frac{3}{4}$. Ask students to describe a simulation using any of the following devices:
    - spinner

      Create a spinner with 4 equal sections. Three of the sections represent a correct prediction and one section represents an incorrect prediction.

    - number cube

      A result of 1, 2, or 3 represents a correct prediction. A result of 4 represents an incorrect prediction. A result of 5 or 6 is ignored and is rolled again.

    - coins

      Flip two coins. A result of HH, HT or TH represents a correct prediction and a result of TT represents an incorrect prediction.

    - random digit generator

      Use digits in pairs. A result from 00 to 74 represents a correct prediction. A result from 75 to 99 represents an incorrect prediction.

    - drawing objects (tiles, marbles, slips of paper) from a bag

      Put three blue marbles and one red marble into the bag. Drawing a blue marble represents a correct prediction. Drawing a red marble represents an incorrect prediction. The marble should be put back into the bag before repeating the simulation.

- Ask students to use a probability device or a random number generator to estimate a probability of a compound event. For example, a soccer player kicks shots against a goalie until the player scores two goals. The soccer player scores goals on 55% of kicks. What is the probability that the player gets the second goal on exactly the third kick? Ask students to use a random number generator to create a list of whole numbers from 1 to 100. Let each number from 1 to 55 represent a scored goal and each number from 56 to 100 represent a failed goal. Suppose that the random number generator gives the following list of digits.

  92, 52, 87, 3, 15, 31, 69, 53, 28, 41, 43

  Based on this list, the first kick fails, the second kick scores, the third kick fails, and the fourth kick scores. Since it took four kicks to score twice, the students record one result that required four kicks. The simulation now starts over. Continuing with the list, the first kick scores and the second kick scores. Since it took two kicks to score twice, the student records a result that required two kicks. These numbers translate into the following kick sequence.

  fail–score–fail–score; score–score; fail–score–score; score–score

  Ask students to repeat this several times. Suppose that in the end the player used three kicks 14 times and did not use three kicks 36 times. Because the player used exactly three kicks 14 times out of 50 trials, estimate that the probability that the player uses exactly three kicks is $\frac{14}{50}$.

Key Academic Terms:

compound event, simulation, frequency, ratio

Additional Resources:

- Lesson: Simulations—Can you design an experiment?
- Activity: Red, Green, or Blue?

16c

Data Analysis, Statistics, and Probability
Investigate probability models.
16. Find probabilities of simple and compound events through experimentation or simulation and by analyzing the sample space, representing the probabilities as percents, decimals, or fractions. c. Represent events described in everyday language in terms of outcomes in the sample space which composed the event.

Guiding Questions with Connections to Mathematical Practices:

How can the probability of a compound event be expressed as a value?

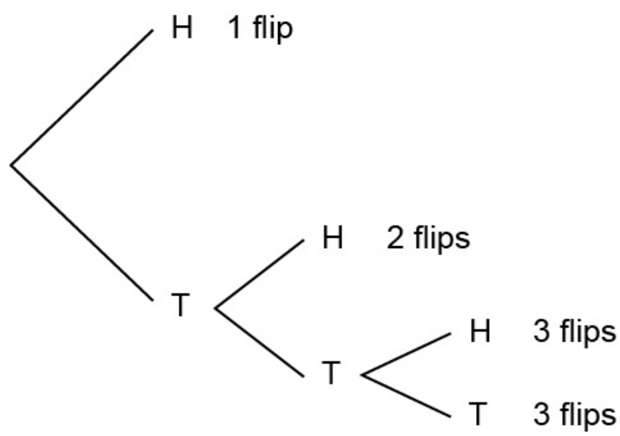
M.P.2. Reason abstractly and quantitatively. Observe that the probability of a compound event is a ratio of the number of favorable outcomes to the total number of possible outcomes. For example, the outcome of getting heads on a coin flip and then rolling a 4 on a number cube is $\frac{1}{12}$, because there is one way to get the favorable outcome of "heads, 4," and there are 12 possible outcomes. Additionally, observe that some models of compound events need to be adjusted in order to yield equally likely outcomes.

- Ask students to find the sample space of a given compound event and use the sample space to find the probability of the compound event. For example, a bag contains six tiles numbered from 1 to 6. Two tiles are randomly drawn. What is the probability that the sum of the tiles is 6? Ask students to choose a representation to help them list the sample space. A possible organized list is shown.

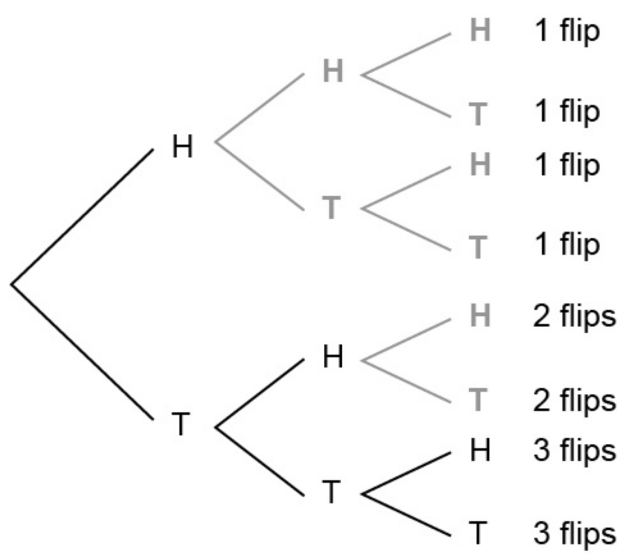
  1, 2 1, 3 1, 4 1, 5 1, 6
  2, 3 2, 4 2, 5 2, 6
  3, 4 3, 5 3, 6
  4, 5 4, 6
  5, 6

  Of the 15 equally likely outcomes in the sample space, 2 of the outcomes have a sum of 6. Therefore, the probability of drawing a sum of 6 is $\frac{2}{15}$.

- Help students to know that models of compound events may need to be adjusted in order to create equally likely outcomes. For example, a student keeps flipping a coin until heads is flipped or until the coin has been flipped three times. What is the probability that the student stops after the third flip? A basic tree diagram is shown.



It appears that there are 2 outcomes out of 4 in which the student stops after three flips. However, not all the outcomes are equally likely. In order to create equally likely outcomes, some placeholder flips need to be added.



With the placeholder flips included, the outcomes are now equally likely. There are 2 favorable outcomes out of a total of 8 outcomes. Therefore, the probability of stopping after three flips is $\frac{2}{8}$ or $\frac{1}{4}$.

Key Academic Terms:

compound event, sample space, tree diagram, organized list, table, ratio, percent, fraction, decimal

Additional Resources:

- Lessons: MAFS.7.SP.3.8
- Video: Menu toss-up: choice and data sets
- Lesson: Compound probability
- Activity: Thinkport | Probability and tree diagrams
- Activity: Tetrahedral dice

17

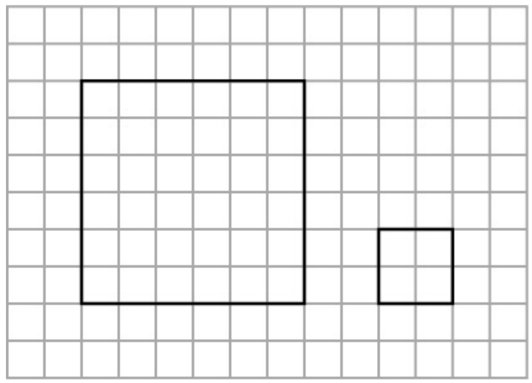
Geometry and Measurement
Construct and describe geometric figures, analyzing relationships among them.
17. Solve problems involving scale drawings of geometric figures, including computation of actual lengths and areas from a scale drawing and reproduction of a scale drawing at a different scale.

Guiding Questions with Connections to Mathematical Practices:

How do scale drawings compare to their original counterparts?

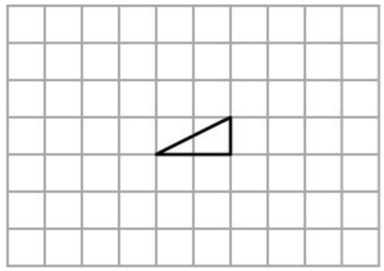
M.P.4. Model with mathematics. Observe that angle measures are preserved in scale drawings and that lengths are changed by a constant ratio called the scale factor. For example, a rectangle with a length of 8 inches and a width of 7 inches could have a scale drawing constructed with a scale factor of $\frac{1}{2}$. The scale drawing would have all right angles with a length of 4 inches and a width of 3.5 inches. Additionally, an equilateral triangle with side lengths 20 centimeters could have a scale drawing constructed with a scale factor of $\frac{3}{5}$. The scale drawing would have all 60° angles with side lengths of 12 centimeters. Further, a regular hexagon with side lengths of 2 inches could have a scale drawing constructed with a scale factor of 6. The scale drawing would have all 120° angles with side lengths of 12 inches.

- Ask students to draw a scale drawing of a given geometric figure with a specific scale factor on graph paper. For example, start with a square with side lengths of 6 units and have the students scale it by a scale factor of $\frac{1}{3}$. Since $\frac{1}{3} \bullet 6 = 2$, the students should draw a square with side lengths of 2 units.

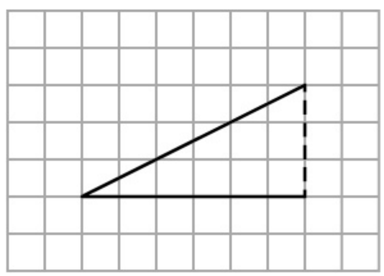


Discuss key features of the two drawings, specifically that the angle measures and shapes are the same and the side lengths changed by the same scale factor.

- Ask students to draw a figure and scale all but one side with a specific scale factor. Then, observe that the last side automatically gets scaled by that same factor just by completing the drawing. For example, give students the figure shown.



Ask students to enlarge the bottom side by a scale factor of three. The original figure has a length of 2 units, so the scale drawing will have a horizontal segment with a length of 6 units. Next ask students to enlarge the slanted side by a scale factor of three. Because it slants right 2 units and up 1 unit, the line drawn should slant to the right 6 units and up 3 units. Note that drawing in the missing segment of their triangle automatically creates a segment that has been enlarged by a scale factor of 3 even though the students did not explicitly calculate it.



In addition, observe that the first triangle has an area of 1 square unit ($A = \frac{1}{2} \times 1 \times 2$) and the second triangle has an area of 9 square units, which does not match the scale factor of three. This is explained by looking at how the area of the second triangle is calculated. The area of the second triangle is $A = \frac{1}{2} \times 3 \times 6$. The 3 and the 6 came from multiplying the original side lengths by 3. So writing the area equation without calculating these products results in $A = \frac{1}{2} \times (1 \times 3) \times (2 \times 3)$, which has the same factors as the area of the small triangle, with the addition of being multiplied by 3 twice. Point out to students that the scaling properties for side lengths do not apply to areas.

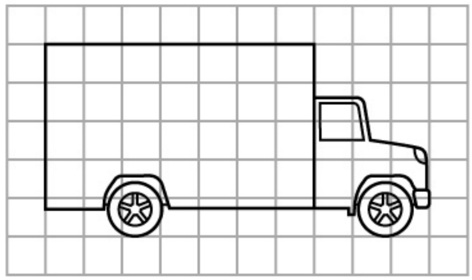
M.P.4. Model with mathematics. Analyze two- and three-dimensional drawings and models to determine if they are made to scale. For example, a drawing of a skyscraper is not to scale if the ratio of actual length to drawing length of one dimension is different than the ratio of actual length to drawing length of another dimension. Additionally, when making a three-dimensional model of a vehicle, all angle measures in the model must be the same as the actual angle measures of the original vehicle.

- Ask students what information is needed to build a three-dimensional model of the classroom. Some possible student responses may include:
    - actual measurements
    - scale factor
    - model measurements
    - materials

  Guide students to find an appropriate scale factor. The scale factor should make a model smaller than the actual classroom, but not so small that the model is difficult to construct. Have students find the model measurements for various objects in the classroom. Start with measuring the objects and then make two-dimensional drawings in the chosen scale factor. Then, try measuring objects and making three-dimensional drawings or models. For example, a classroom that has a width of 29 feet, a length of 32 feet, and a height of 18 feet could use a scale factor of 1 inch equals 2 feet to make a model that is 14.5 inches wide, 16 inches long, and 9 inches tall. Additionally, a student desk in the classroom that is 1.5 feet wide, 1.5 feet long, and 2.5 feet tall would be represented by a model that is 0.75 inches wide, 0.75 inches long, and 1.25 inches tall.

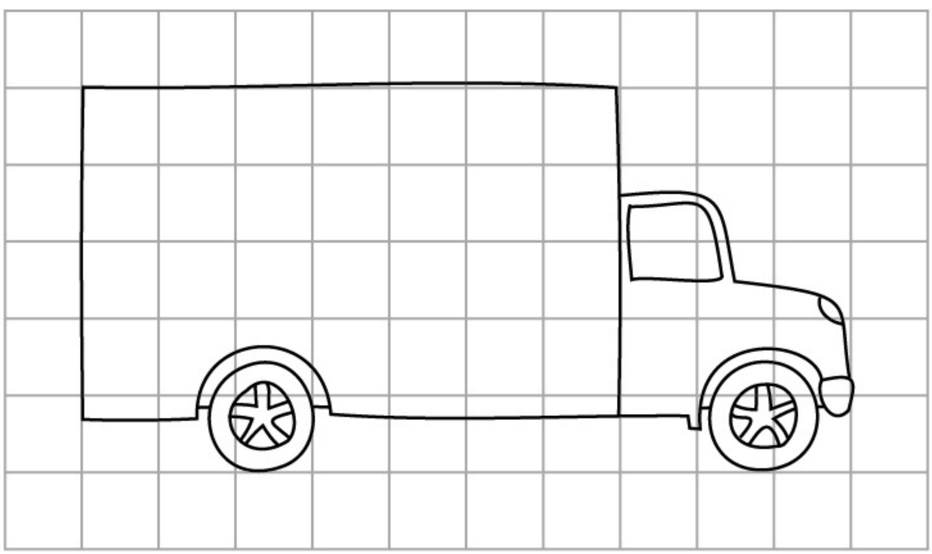
- Ask students to create a scale drawing of an image by putting the image on a grid and using the grid to scale the image up or down. For example, the students could make a scale drawing of the image of the truck shown using the scale factor of 2.

Original Image

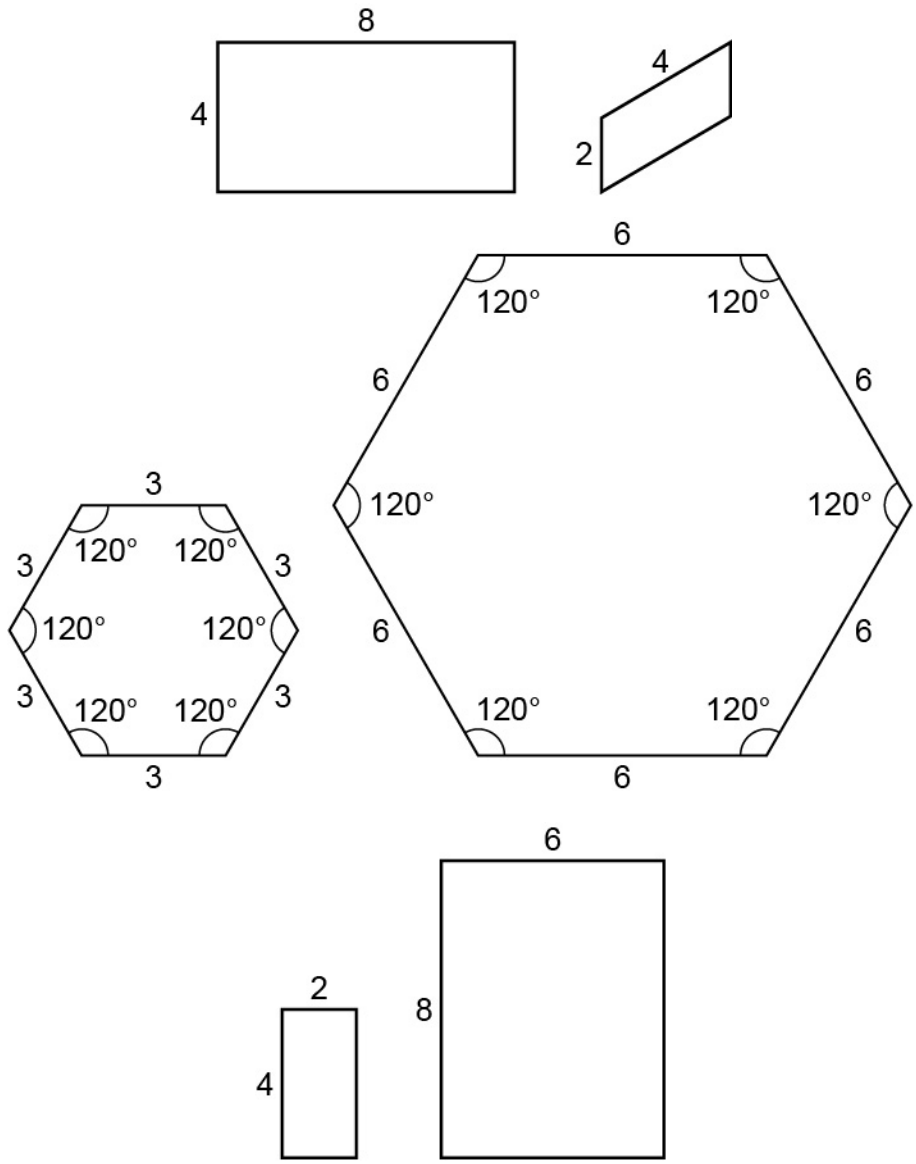


This can be done by making a grid on a piece of paper with squares that have side lengths twice the length of the squares on the original grid. Then, the students draw each square from the original image into the new, larger grid.

Scale Image



- Ask students to compare two figures and determine whether one is a scale drawing of the other. Some example pairs of figures are shown.

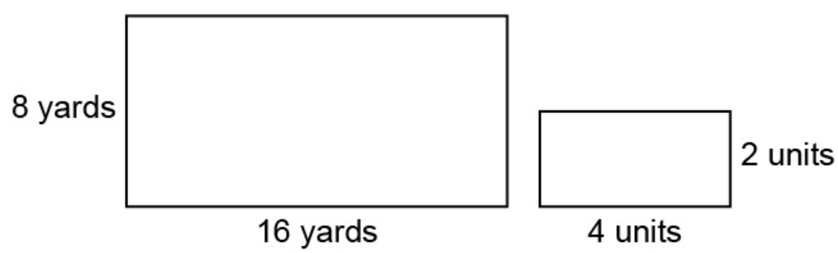


In the given examples, the hexagons are the only scale drawings because the shape and angle measures are maintained and the side lengths change by a common scale factor. The other two examples either do not maintain angle measures or do not have a common scale factor.

How does scale factor affect the area of a scale drawing?

M.P.6. Attend to precision. Observe that all the dimensions in a geometric figure are multiplied by a scale factor to create a scale drawing and that the area of the scale drawing will change by the scale factor squared because the scale factor is used as a factor twice, once for each dimension of the square unit that is used to measure area. For example, a square rug with side lengths of 10 feet may have a scale drawing constructed using 1 unit = 2 feet so that the scale drawing is 5 units on each side. The area of the scale drawing is 25 square units, and the area of the original rug is 100 square feet, which is equivalent to the area of the scale drawing multiplied by the scale factor squared ($25 \bullet 2^2 = 100$). Additionally, a circular window has a radius of 12 inches. A scale drawing of the window has 1 unit = 3 inches so that the scale drawing has a radius of 4 units. The area of the scale drawing is 16π square units and the area of the window is 144π square inches. The area of the scale drawing is equivalent to the area of the window divided by the scale factor squared ($144\pi \div 3^2 = 16\pi$).

- Ask students to find the scale factor given the area of a scale drawing and the area of the original figure by making multiplicative comparisons of the areas. For example, the area of a rectangle is 128 square yards, and the scale drawing of the rectangle has an area of 8 square units. The original rectangle's area is 16 times as great as the area of the scale drawing, so the square of the scale factor is 16. Since $4 \bullet 4 = 16$, the scale used in this example is 4 yards = 1 unit.



- Ask students to find the area of an original figure given a scale factor and the area of a scale drawing. For example, a scale drawing of a circle has an area of 9π square units and the scale is 1 unit = 7 meters. Since the scale factor is 7, the area of the original circle must be 7^2 times as great. The area of the original circle is $9\pi \bullet 49$, or 441π square meters.

- Ask students to find the area of a scale drawing based on the area of a real-world object and to confirm the area by constructing a scale drawing of the object. For example, a billboard on the side of a highway has dimensions of 14 feet by 48 feet. The area of the billboard is covered in 672 square feet of vinyl. Construct a scale drawing on poster board that is 24 inches wide. The height of the poster board needs to be 7 inches to maintain the same scale of 2 feet = 1 inch. The area of the scale drawing is $7 \bullet 24 = 168$ square inches. Verify that the scale factor is 2 and that the area of the scale drawing is 672 divided by 2^2, or $672 \div 4 = 168$.

Key Academic Terms:

scale drawings, geometrical figures, area, scale factor, ratio, proportion, scale up, scale down, multiplicative comparison

Additional Resources:

- Activity: Floor plan
- Activity: Rescaling Washington Park
- Lesson: Scale drawings
- Activity: Tinkercad

18

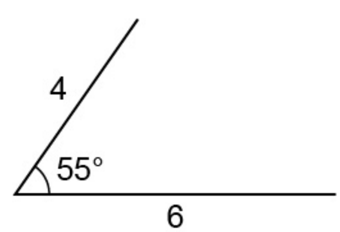
Geometry and Measurement
Construct and describe geometric figures, analyzing relationships among them.
18. Construct geometric shapes (freehand, using a ruler and a protractor, and using technology), given a written description or measurement constraints with an emphasis on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Guiding Questions with Connections to Mathematical Practices:

How can tools be used to draw geometric shapes with given conditions?

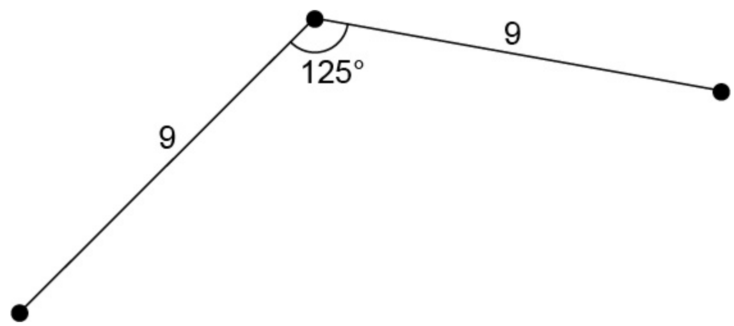
M.P.5. Use appropriate tools strategically. Use tools like a ruler, protractor, and/or technology to draw shapes and manipulate drawings to match given conditions. For example, use a protractor and ruler to draw a 75° angle with 4-inch line segments. Then, draw a third line segment to close the figure and create an isosceles triangle. Additionally, use an online geometry tool to draw a 60° angle with equal line segments constructing the angle and then draw a third line segment to complete an equilateral triangle.

- Ask students to choose an angle measure and two side lengths that form the sides of the angle for the start of a triangle drawing and construct it using a ruler and a protractor. For example, students might pick an angle measure of 55° and side lengths of 4 and 6 centimeters.

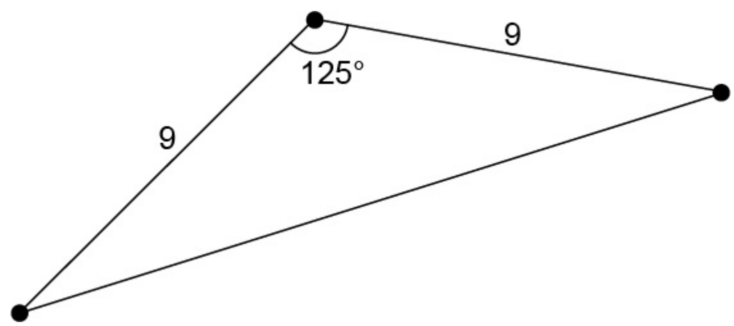


Complete the triangle by adding the third line segment. This will demonstrate that given one angle measure and the two side lengths that construct the given angle, there is only one possible side length for the third side. Have students use a ruler to verify.

- Ask students to construct an obtuse angle using legs of the same length with an online geometry tool. For example, an angle is shown that measures 125° and has side lengths of 9 units.



Connect the two side lengths to create an obtuse isosceles triangle.

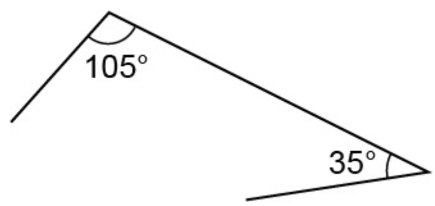


- Ask students to spend some time exploring an online geometry tool, using the measurement features to see how angle measure and side lengths interact for various geometric shapes, with special attention to triangles.

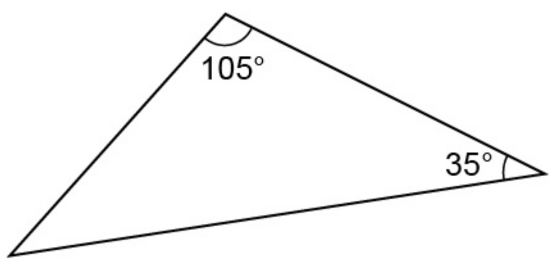
How can the properties of triangles be used to generalize the conditions when there is one unique triangle, more than one triangle, or no triangle?

M.P.7. Look for and make use of structure. Observe that when two angles in a triangle are given, the third angle is a forced measurement, leading to many possible triangles that all have the same angle measurements. Two triangles that have all the same angle measurements are similar. For example, use a protractor and ruler to draw several triangles that have a 40° angle and a 30° angle. Then, observe that the third angle in all the triangles has a measure of 110°. Additionally, use an online geometry tool to draw several triangles that have a 70° angle and a 20° angle. Then, observe that the third angle in all the triangles has a measure of 90°.

- Ask students to construct two angles with given measurements (that have a sum of less than 180). Ask students to construct the angles at the endpoints of a single line segment so that both angles lie on the same side of the segment. For example, a triangle is shown with the angle measures 105° and 35°.

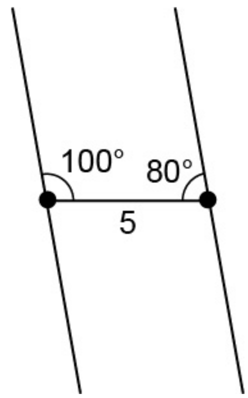


Complete the triangle by extending the two side lengths until they meet.



Compare triangles amongst the students. All three angles being of equal measure does not mean that all the triangles will be identical. Have students measure side lengths to verify.

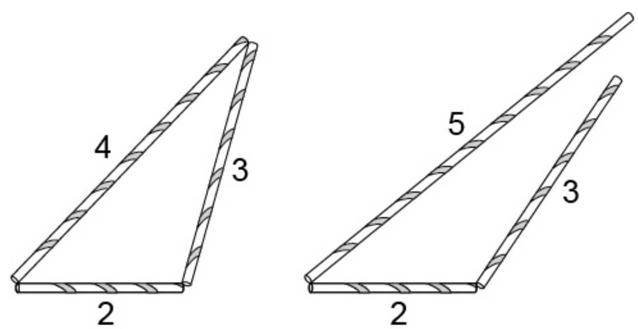
- Using an online geometry tool, ask students to construct a triangle given two angle measures and the length of the side between the angles. For example, a line segment is shown that is 5 units in length with an angle that measures 100° on one endpoint and an angle that measures 80° on the other endpoint.



Students can then explore whether a triangle with these angle measures is possible. Students should see informally that the line segments on either side of the 5-unit segment are parallel, making constructing a triangle impossible. Students may also notice that the three angle measures of triangles always sum to 180°, so having an angle of 100° and another of 80° will not make a triangle. Ask students to predict what changing the 80° to 79° does to the diagram. To demonstrate the change in the online geometry tool, zooming out the view of the figure may be required.

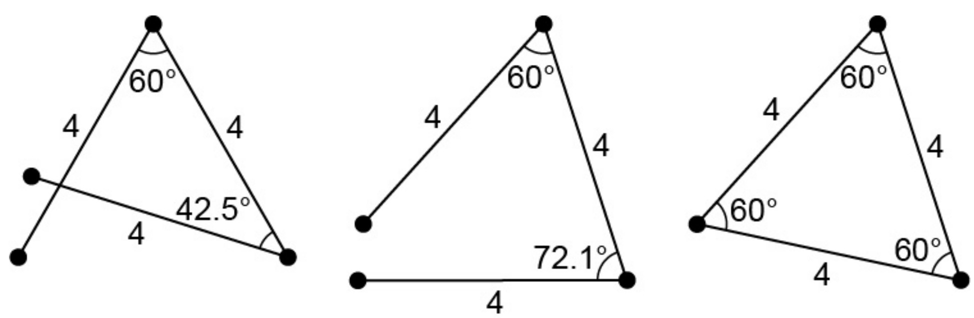
M.P.1. Make sense of problems and persevere in solving them. Construct a triangle with given side lengths to make a unique triangle and informally generalize that in order to be a triangle, the sum of side a and side b must be greater than side c. For example, line segments with measurements of 4 inches, 5 inches, and 10 inches will not make a triangle because 10 is larger than $4 + 5$. Additionally, when all three side lengths are given, there is only one possible way to make a triangle, even though the orientation of the sides may differ.

- Ask the students to construct triangles with drinking straws of varying lengths. For example, provide straws of lengths 2, 3, 4, 5, and 6 inches. Experiment with combinations of lengths that do and do not make triangles. For example, side lengths of 2, 3, and 4 inches make a triangle, but side lengths of 2, 3, and 5 do not make a triangle.



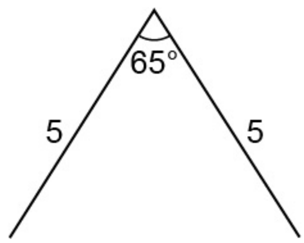
Discuss which combinations of straws create triangles. Make the connection that the longest straw is always shorter than the sum length of the other two straws.

- Ask students to construct triangles using only popsicle sticks of equal length. Use a protractor to measure the angles of the triangle, which should all be 60°. Explore triangles with varying lengths of sticks, straws, or other manipulatives, always using three equal-length sides. Continue to measure the angles with a protractor. Then, ask students to attempt to make a triangle with equal side lengths that does not have all 60° angles. Confirm that it is not possible using an online geometry tool.

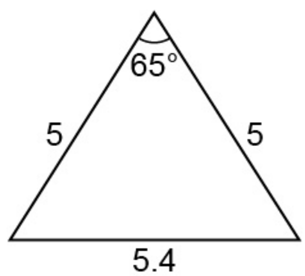


M.P.7. Look for and make use of structure. Observe that given two side lengths and the angle measure between them, exactly one triangle can be created. For example, use technology to construct a 65° angle with side lengths of 4 units and 5 units and then observe that there is only one possible way to construct the third side. Therefore, all triangles with this construction are congruent. Additionally, note that it is not enough to know that the angle measures of two triangles are equivalent to show that the triangles are the same—at least one pair of corresponding side lengths must also be equal.

- Ask students to construct triangles with a given angle measure and two equal side lengths. For example, a 65° angle is shown with 5-inch line segments.

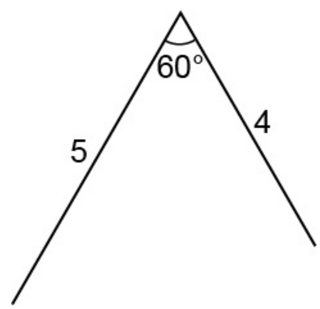
  

  Complete the triangle by connecting the two ends and measure the third side length using a ruler.

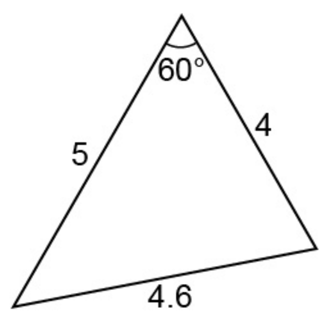
  

  Demonstrate that there is only one possible side length for the third side of the triangle, making the triangle unique. Also, clarify that even though the first two side lengths are equal lengths, the triangle is not automatically equilateral.

- Ask the students to construct triangles beginning with a 60° angle and sides of different lengths. For example, a side of 5 centimeters and a side of 4 centimeters with a 60° angle are shown.



Connect the two sides to complete the triangle and measure the third side length using a ruler.

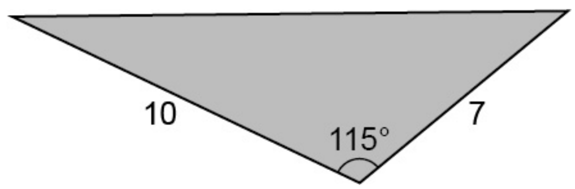


Draw attention to the angle measure of 60° and the different side lengths, clarifying that having one 60° angle does not mean that the triangle is equilateral.

What does it mean for a triangle to be unique?

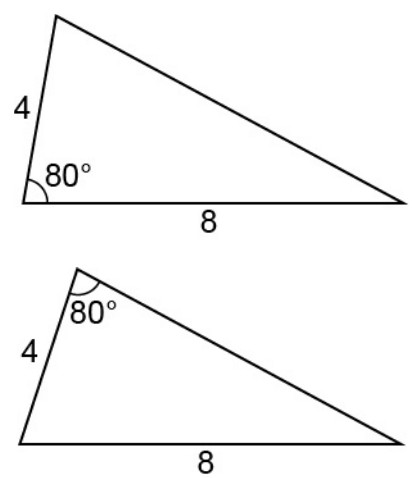
M.P.6. Attend to precision. Identify that two triangles are the same if one can be flipped, rotated, or moved to match the other triangle. For example, a triangle with side lengths 1.5 units, 2 units, and 3 units oriented with the 3-unit side facing south is the same triangle as one with side lengths 1.5 units, 2 units, and 3 units oriented with the 1.5-unit side facing south. Additionally, the two triangles will have equal angle measures. Further, two triangles with angle measures that are equal are not necessarily the same triangle, because the side lengths could be different.

- Ask students to cut a triangle out of paper with given measurements. For example, a triangle is shown with an angle that measures 115° and sides of 7 and 10 centimeters on either side of the angle.

  Ask students to compare their triangles. Explore what happens when the triangles are rotated and flipped. Discuss how the triangles are the same and show that they line up with each other when oriented in the same way.

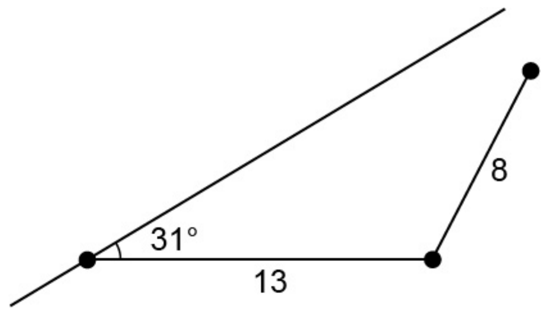
- Ask students to create two triangles with a given angle measure and two side lengths. For example, the triangle may have an angle that measures 80° and side lengths of 4 and 8 inches. Two possible triangles are shown.

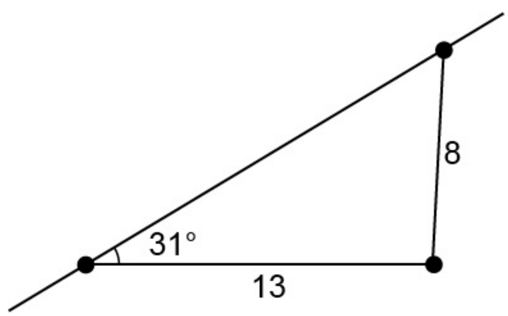
  Discuss whether the triangles are unique or the same. Use a ruler and protractor to measure the other angles and side length to verify.

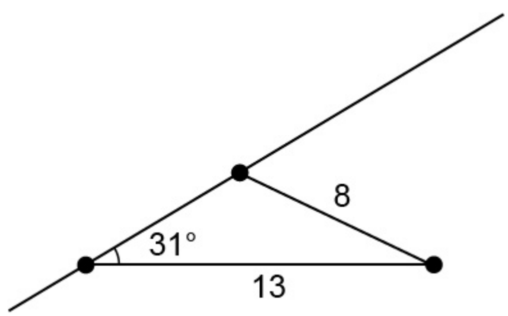
M.P.7. Look for and make use of structure. Explore special cases of triangles, such as the conditions when two unique triangles are created. For example, given the requirements that two of the side lengths of a triangle must be 5 and 3 and that one angle that is not formed by the sides with lengths 5 and 3 must be 20°, two unique triangles can be created. Additionally, given two side lengths and no set angle measure, many triangles can be created.

- Ask students to use an online geometry tool to create the first two sides of a triangle with a given angle measurement and two given side lengths. For example, create an angle with a measure of 31° and a segment of 13 units on one side. On the other endpoint of the 13-unit side, create a segment of 8 units, with no fixed angle measurement, as shown in the figure.



Ask students to experiment and find how many triangles can be made by moving the 8-unit leg. Exactly two triangles are possible.





Discuss what information is needed to determine whether two triangles are unique. It is not enough to know two sides and an angle are equivalent unless the known angle measure is between the two known side lengths.

- Ask students to construct triangles given two side lengths and no set angle measures. For example, construct triangles with sides of 6 inches and 3 inches. Students then complete their triangles and use a ruler and protractor to measure the remaining side lengths and all angles. Students can then compare triangles and note that there are many possible ways to construct a triangle given only two side lengths.

Key Academic Terms:

geometric shapes, angles, triangles, conditions, construct, protractor, leg, polygon

Additional Resources:

- Activity: A task related to standard 7.G.A.2
- Lesson: Grade 7 mathematics module 6, topic B, lesson 6
- Lesson: Triangle inequality theorem
- Activity: Triangle maker

19

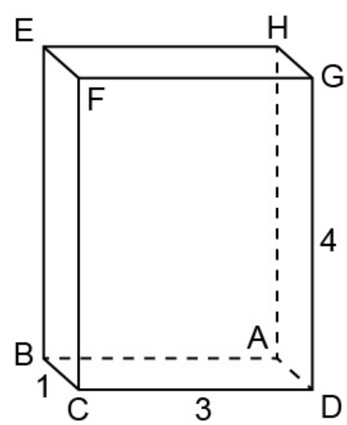
Geometry and Measurement
Construct and describe geometric figures, analyzing relationships among them.
19. Describe the two-dimensional figures created by slicing three-dimensional figures into plane sections.

Guiding Questions with Connections to Mathematical Practices:

How can slicing a three-dimensional figure with a two-dimensional plane section help to understand the composition of the figure?

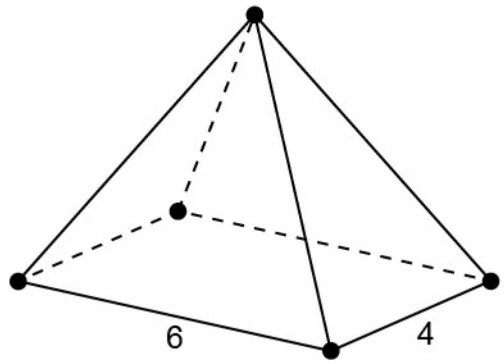
M.P.7. Look for and make use of structure. Observe that right rectangular prisms and right rectangular pyramids are composed of two-dimensional horizontal slices of rectangles stacked to a specified height. For example, a right rectangular prism can be sliced parallel to its base to make a rectangle with dimensions equal to its base. Additionally, a right rectangular pyramid can be sliced parallel to its base to make a rectangle similar to its base.

- Ask students to find the area and perimeter of different cross sections of a right rectangular prism, keeping the cross sections parallel to one face of the prism. For example, shown is a right rectangular prism with a length of 3 units, a width of 1 unit, and a height of 4 units.

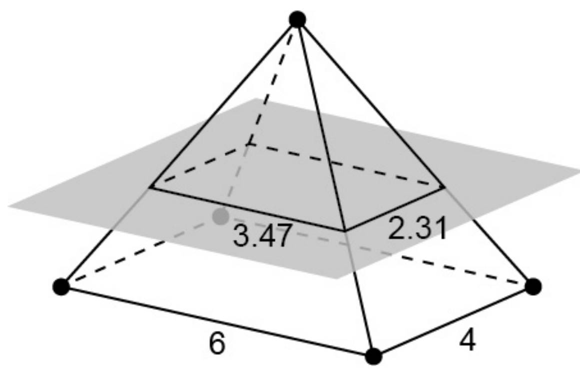


Any cross section that is parallel to the base has a perimeter of 8 units and an area of 3 square units, just like the base. However, when a cross section is made that is parallel to the face BCFE, that cross section has a perimeter of 10 units and an area of 4 square units, just like face BCFE. And similarly, a cross section parallel to face CDGF has a perimeter of 14 units and an area of 12 square units, just like face CDGF.

- Ask students to create a right rectangular pyramid with given dimensions using an online geometry tool. For example, shown is a right rectangular pyramid with a height of 5 units and a base with a length of 6 units and a width of 4 units.



Create a plane intersecting the right rectangular pyramid parallel to the base. Use the online geometry tool to find the dimensions of the rectangular cross section created by the plane.



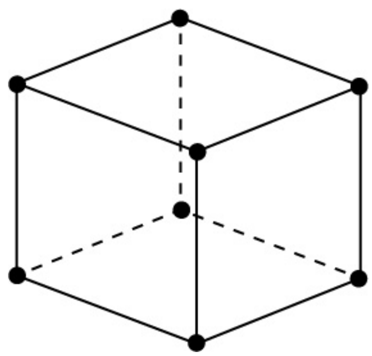
To find the ratio of proportionality, use the dimensions of the cross section and make a ratio with the dimensions of the original base. For example, $\frac{3.47}{6}$ and $\frac{2.31}{4}$ are the ratios for the length and width in this example. Both ratios are equal to 0.58 when rounded to the nearest hundredth.

How can the properties of two- and three-dimensional figures be used to identify what three-dimensional object a given plane section represents?

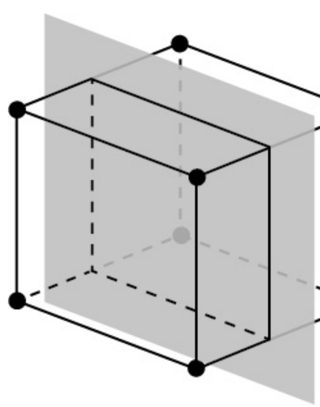
M.P.4. Model with mathematics. Explore plane sections of three-dimensional figures to identify the possible two-dimensional shapes that are created. For example, slice a cube in a variety of ways to show that a cube has plane sections in the shape of squares, rectangles, trapezoids, parallelograms, triangles, pentagons, and hexagons. Additionally, slice a right rectangular pyramid in a variety of ways to show that a right rectangular pyramid has plane sections in the shape of triangles, rectangles, trapezoids, and pentagons.

- Ask students to create right rectangular prisms with a manipulative like modeling clay. Have students make predictions about what two-dimensional shapes can be made by slicing the prisms in different ways. Then, have students slice the prisms to test the predictions. Discuss which shapes are possible and which are not.

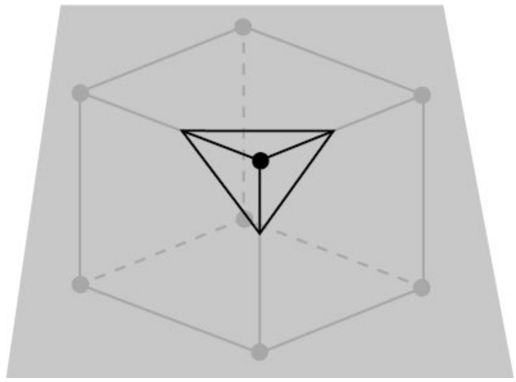
- Ask the students to create a cube using an online geometry tool.



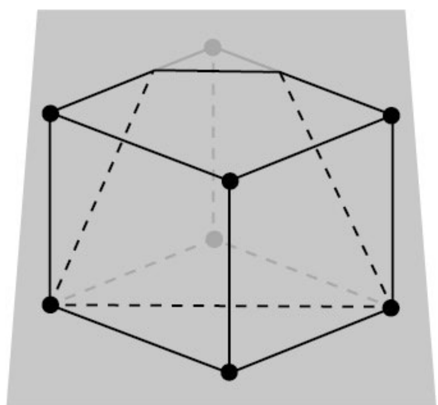
Have students use an intersecting plane to create a square cross section.

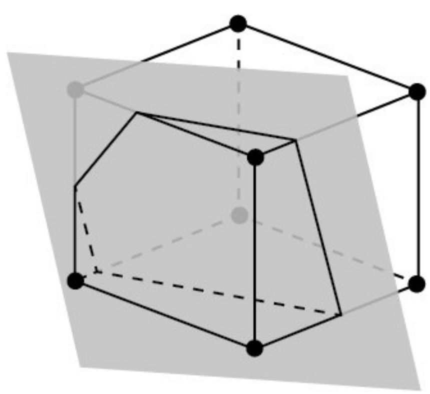


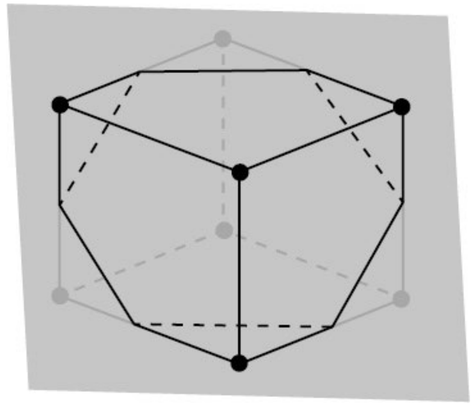
Discuss that the plane must be parallel to one face of the cube in order to make a square cross section. Next, have the students use an intersecting plane to make a cross section that is a triangle. One possibility is shown.



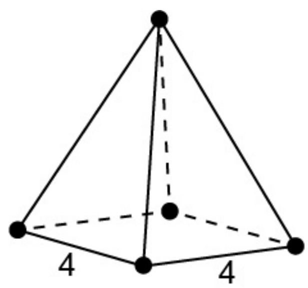
Discuss that the plane cannot be parallel to any base to make a triangular cross section, and it will only intersect three of the faces of the cube. Finally, allow students to create intersecting planes that make trapezoid-, pentagon-, and hexagon-shaped cross sections.



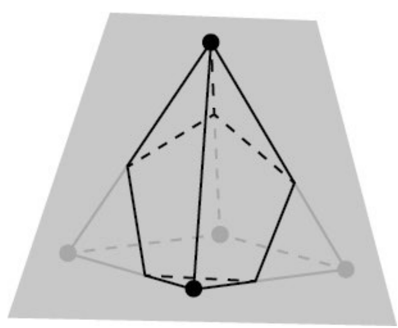




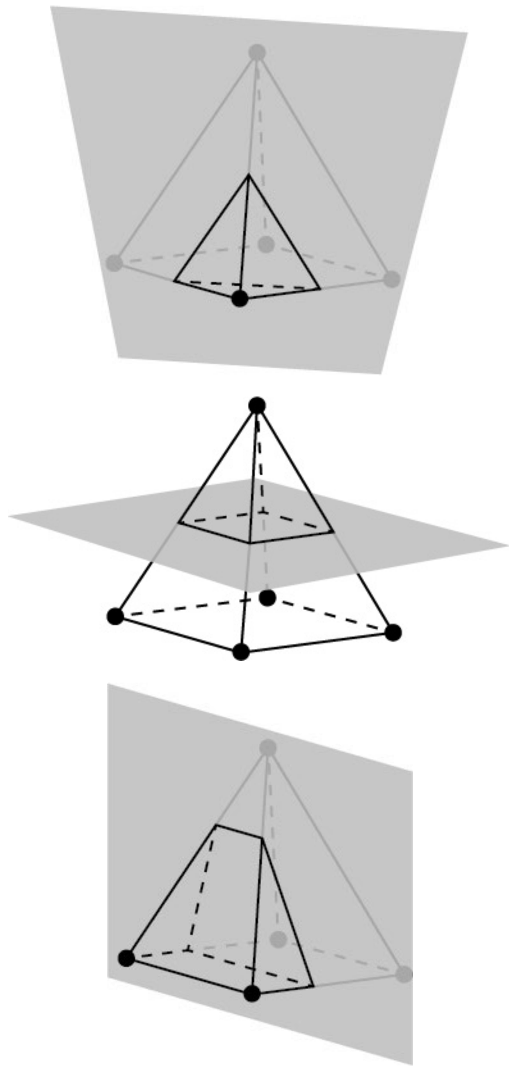
- Ask students to create a square pyramid using an online geometry tool. For example, shown is a square pyramid with a height of 4 units and a square base of 4 units on each side.



Have students experiment with intersecting planes to create a variety of cross section shapes. Record the cross sections made and try to make as many shapes as possible. A possible pentagon-shaped cross section is shown.



Other examples of possible cross sections shapes are triangles, squares, and trapezoids, as shown.



Key Academic Terms:

two-dimensional, three-dimensional, perpendicular, parallel, construct, base, horizontal, vertical, slice, plane section, right rectangular prism, right rectangular pyramid, cross section

Additional Resources:

- Lesson: [Can you cut it? Slicing three-dimensional figures]
- Activity: [Cube ninjas!]
- Video: [Slicing three-dimensional figures]
- Activity: [Cross sections of a cube]

20a

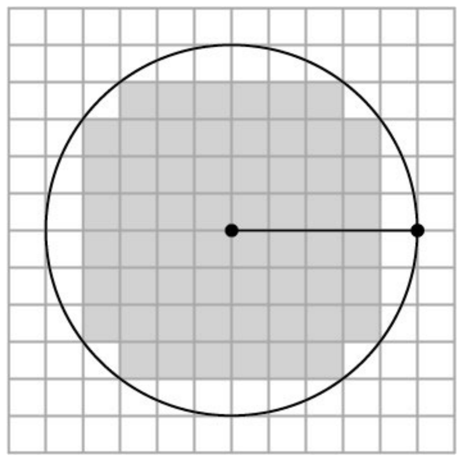
Geometry and Measurement
Solve real-world and mathematical problems involving angle measure, circumference, area, surface area, and volume. *Note: Students must select and use the appropriate unit for the attribute being measured when determining length, area, angle, time, or volume.*
20. Explain the relationships among circumference, diameter, area, and radius of a circle to demonstrate understanding of formulas for the area and circumference of a circle. a. Informally derive the formula for area of a circle.

Guiding Questions with Connections to Mathematical Practices:

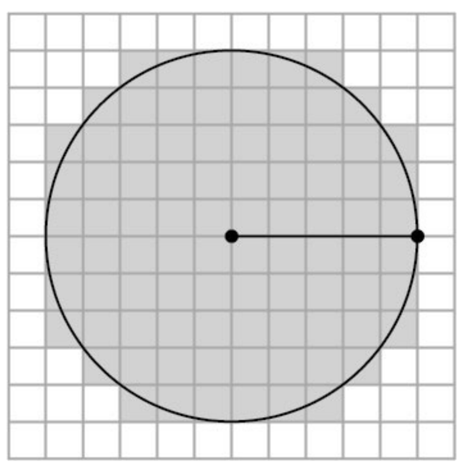
How can the area and circumference of a circle be used to help solve problems?

M.P.4. Model with mathematics. Use the formulas for area and circumference to find missing quantities in problems involving circles. For example, a circle with a circumference of 8π units must have a diameter of 8 units and a radius of 4 units. The area of the circle can be found by using the radius: $4^2 \bullet \pi = 16\pi$ square units. Additionally, a circle with an area of 49π square units must have a radius of 7 units because the square root of 49 is 7. The circumference can be found by using the radius: $2 \bullet \pi \bullet 7 = 14\pi$ units.

- Ask students to estimate the area of a circle given on a grid. The center of the circle should lie on an intersection of the gridlines. Ask students to shade in and count all the complete squares inside the circle. The shaded squares form a shape that is contained entirely inside the circle. Therefore, the area of the shape (the number of shaded squares) is less than the area of the circle. For example, give students a circle with a radius of five units. The shaded region has 60 squares in it, and, therefore, the area of the circle is greater than 60 square units.



Next, ask the students to shade in and count all the squares that overlap the interior of the circle. The shaded squares form a shape that contains the entire circle. Therefore, the area of this new shape is greater than the area of the circle. For example, this shaded region for the circle has 88 squares, so the area of the circle is less than 88 square units.



Then, ask students to use these two pieces of information to estimate the actual area of the circle. The estimate needs to be between 60 and 88 because the area of the circle is greater than the area of the first shape and less than the area of the second shape. Ask students to compare the estimate to the area found when using the area formula, $A = \pi r^2$.

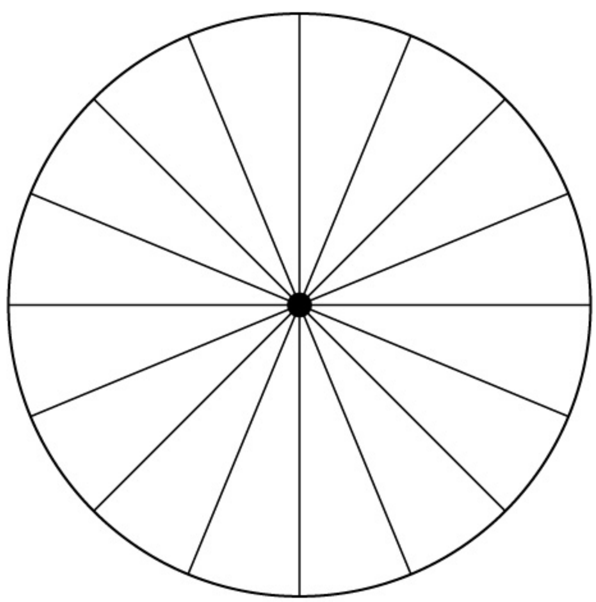
- Ask the students to measure a variety of circular objects using string and a ruler. Discuss how to find the circumference using the diameter with the equation $C = d\pi$. Compare the calculated circumference to the measured circumference. For example, measure the circumference of a saucer and find that it is about 37.8 centimeters. The diameter of the saucer is 11.9 centimeters. Using the formula $C = d\pi$, the circumference should be about 37.47 centimeters. Discuss the difference between the measured circumference and the calculated circumference.

- Ask students to create circles using strings of different lengths so that each student has a circle of different size. Ask students to find the area of the circles by using the length of the strings as the circumferences of the circles. For example, a string that is 10 inches long will make a circle with a circumference of 10 inches. Substitute 10 in for C in the equation $C = 2\pi r$ and use 3.14 to estimate pi. So the equation $10 = 2 \bullet 3.14 \bullet r$ solved for r shows that the radius is about 1.59 inches. The radius is then used in the area formula, so $A = 3.14 \bullet (1.59)^2$ shows that the area is about 7.94 square inches.

How is the circumference related to the area of the circle?

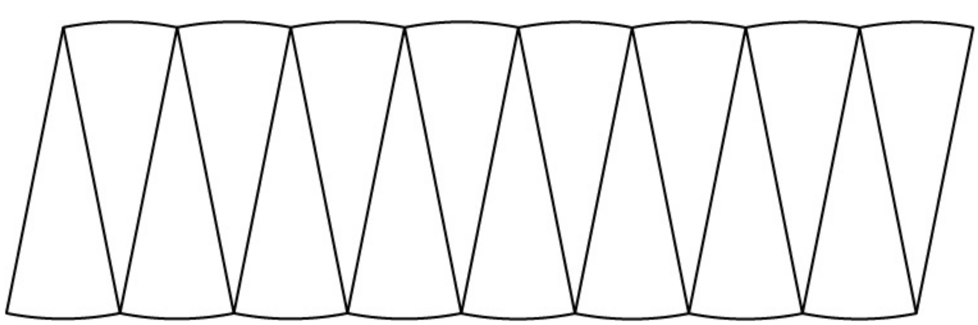
M.P.2. Reason abstractly and quantitatively. Explain the fundamental constant, pi, that applies to all circles and how it is used in the formulas for both area and circumference. For example, a circle with a radius of 3 units has a circumference of 6π units and can be cut into very small slices and rearranged into the semblance of a rectangle with length 3π units and width 3 units. The length of 3π units comes from the circumference 6π in two pieces, one for each long side of the rectangle, and the width of 3 units comes from the radius of the circle. The area of the circle is then estimated to be the length times the width, or 9π square units. Additionally, the circumference and area formulas, $C = 2\pi r$ and $A = \pi r^2$, can be manipulated to show that the circumference and area of a circle are related no matter the size of the circle.

- Give students a circle with the center marked. Ask students to draw a diameter of the circle and record the length of the diameter. Then, continue drawing diameters on the circle to create 16 equal-sized pieces.

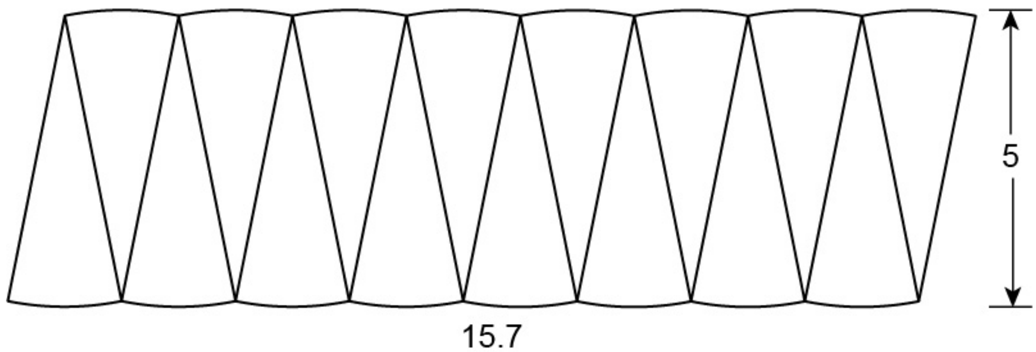


diameter = 10 units

Have students cut out the pieces of the circle and stack the pieces together, alternating the orientation of the pieces, to create an approximate parallelogram.

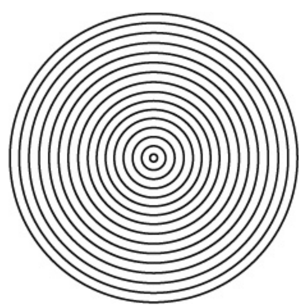


Point out that the circumference of the original circle now makes up the two long sides of the "parallelogram." Discuss why this means that the base of the "parallelogram" has a length of $\pi \bullet r$. Then, look at the height of the "parallelogram" and note that it is the radius of the original circle. For example, if the diameter is 10 units, like in the figure shown, then the radius is 5 units, and the base of the "parallelogram" is about 15.7 units.

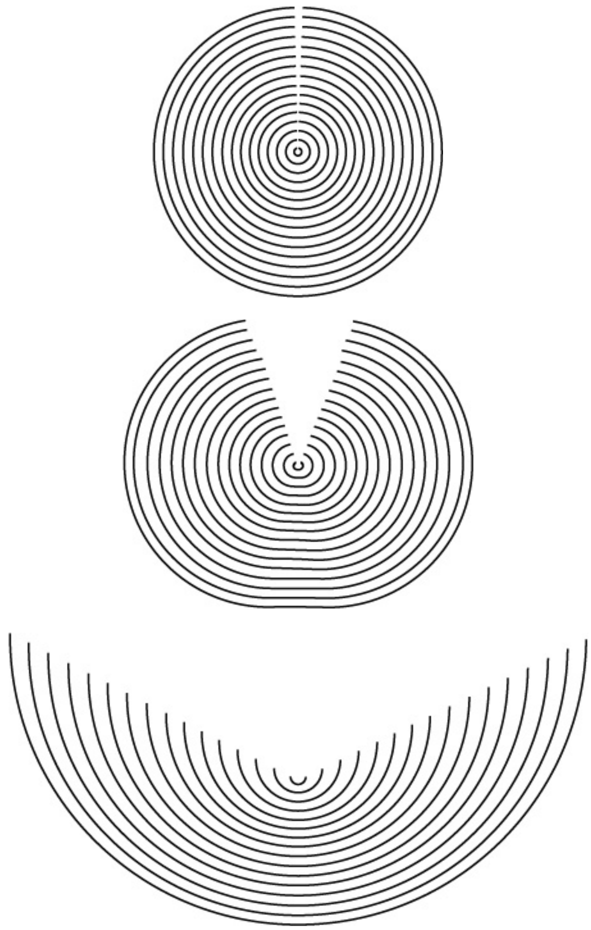


Use this information to examine the area of the "parallelogram" and the area of the original circle. For example, the area of a parallelogram is base times height, so $15.7 \bullet 5 = 78.5$, meaning the area of the "parallelogram" shown is about 78.5 square units. Using the formula for the area of a circle, $3.14 \bullet 5^2 = 78.5$, the area of the circle is also about 78.5 square units. Discuss why this "parallelogram" method works for finding the area of a circle.

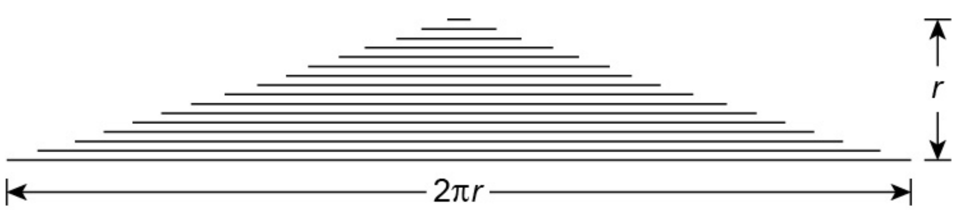
- Show students a series of concentric circles and ask them to imagine that more and more circles can be added until the outer circle is completely filled.



Then, ask students to imagine what would happen if each circle was sliced open and “unrolled” to lay flat.



Show students that the result is approximately a triangle. The base of the “triangle” is made from the circumference of the outermost circle, which means the base is $2 \bullet \pi \bullet r$ units. The height of the “triangle” is the thickness of the layers of all the concentric circles, which is r units. Use the area formula for a triangle to get $A = \frac{1}{2}(r)(2\pi r) = 1 \bullet \pi \bullet r^2 = \pi \bullet r^2$.



Observe that the area formula for the “triangle” yields the area formula for a circle, $A = \pi r^2$.

Key Academic Terms:

circle, area, circumference, radius, diameter, pi (π), derive, unit, square unit

Additional Resources:

- Video: Area of a circle, how to get the formula
- Tutorial: Derivation of the area of a circle
- Activity: Eight circles
- Activity: Designs
- Lesson: Area and circumference in real situations

20b

Geometry and Measurement
Solve real-world and mathematical problems involving angle measure, circumference, area, surface area, and volume. *Note: Students must select and use the appropriate unit for the attribute being measured when determining length, area, angle, time, or volume.*
20. Explain the relationships among circumference, diameter, area, and radius of a circle to demonstrate understanding of formulas for the area and circumference of a circle. b. Solve area and circumference problems in real-world and mathematical situations involving circles.

Guiding Questions with Connections to Mathematical Practices:

How are proportions used to solve problems about the circumference of a circle?

M.P.1. Make sense of problems and persevere in solving them. Identify that the circumference of a circle is proportional to twice its radius, or diameter, multiplied by a constant of pi. For example, a circle with a radius of 3 units has a circumference of 6π units. When the radius is 8 units, the circumference is 16π units, which is the same as the diameter times pi. Additionally, a circle with a diameter of 15 units has a circumference of 15π units.

- Ask students to complete a table by measuring circular objects. A sample table with some possible objects to measure is shown.

Name of Object	Circumference	Diameter	Circumference ÷ Diameter (rounded to the nearest hundredth)
soup can	17.5 cm	5.5 cm	3.18
bowl	22 cm	7 cm	3.14
plate	63 cm	20 cm	3.15
key ring	6 cm	2 cm	3

After measuring the circumference and diameter of each object, ask students to calculate the ratio of the circumference to diameter and record the result. Discuss the values of the ratio columns, noting that this ratio is the definition of pi. Remind students that 3.14 is just an approximation for pi, so even with perfect measurements, the ratio will not be equal to 3.14, as pi is not exactly equal to 3.14.

- Observe that for any two circles, the proportion $\frac{C_1}{d_1} = \frac{C_2}{d_2}$ is true because both ratios are equal to pi. Ask students to use this fact to calculate the circumference of a circle without using pi given the diameter of the circle and the measurements of another circle. For example, tell students that one circle has a diameter of about 10.6 centimeters and a circumference of about 33.3 centimeters. Ask them to use this information to find the circumference of a circle with a diameter of 4 centimeters by setting up the proportion $\frac{33.3}{10.6} = \frac{C_2}{4}$ and solving for C_2. In this case, the result is a circumference of about 12.6 centimeters.

How can problems involving the area and circumference of a circle be solved?

M.P.1. Make sense of problems and persevere in solving them. Use the relationships between circumference, radius, diameter, and area of a circle to solve real-world and mathematical problems. For example, if the diameter of a bicycle tire is 26 inches, the circumference of the tire can be found by substituting 26 for d in the formula $C = \pi d$, where C is the circumference. Additionally, knowing the diameter of a circle leads to knowing the radius, which is equal to half the diameter. That information can be used to find the area of a circle using the formula $A = \pi r^2$, where r is the radius.

- Ask students to solve a problem where the circumference or diameter of a circle is given and the area needs to be found. For example, the diameter of a circular maintenance hole cover in the middle of a street is 60 centimeters. To find the area, students first need to find the radius. In this case, the radius is 30 centimeters because it is half of the diameter. Next, the formula $A = \pi r^2$ can be used, with 30 substituted for r:

$$A = \pi \bullet 30^2$$

$$A = 900\pi$$

Using 3.14 as an approximation for pi, the area is about 2,826 square centimeters.

Key Academic Terms:

circle, area, circumference, radius, diameter, pi (π), derive, unit, square unit

Additional Resources:

- Video: Area of a circle, how to get the formula
- Tutorial: Derivation of the area of a circle
- Activity: Eight circles
- Activity: Designs
- Lesson: Area and circumference in real situations

21

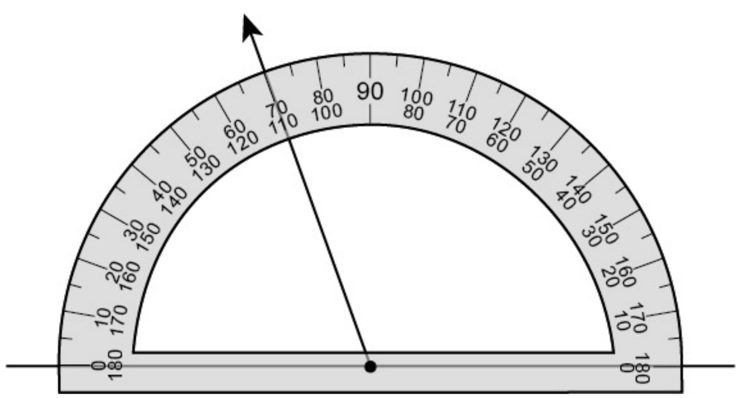
Geometry and Measurement
Solve real-world and mathematical problems involving angle measure, circumference, area, surface area, and volume. *Note: Students must select and use the appropriate unit for the attribute being measured when determining length, area, angle, time, or volume.*
21. Use facts about supplementary, complementary, vertical, and adjacent angles in multi-step problems to write and solve simple equations for an unknown angle in a figure.

Guiding Questions with Connections to Mathematical Practices:

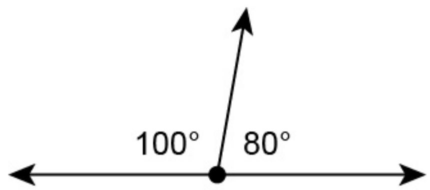
What are supplementary angles and how are they related to each other?

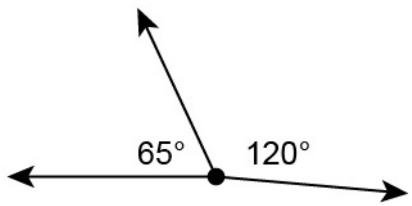
M.P.7. Look for and make use of structure. Know that two angles that can be put together to form a line are called supplementary and that their angle measures sum to 180°. For example, a figure with a horizontal line that has a ray extended from the middle at a 35° angle creates a supplementary angle of 145°. Additionally, the measure of an angle supplementary to 103° can be determined by finding the difference of 180° and 103°.

- Ask students to determine the sum of the measures of angles created when a ray intersects a line. For example, draw a ray from the middle of a given line, and using a protractor, find the measures of the angles to be 70° and 110°, which sum to 180°.



- Ask students to explain why a selected pair of angles does or does not create a supplementary pair. For example, given angles with measures of 100° and 80°, determine that they are supplementary by arranging them to create a straight line, or given angles with measures of 120° and 65°, determine that they are not supplementary.



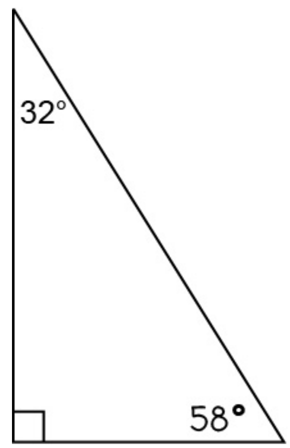


What are complementary angles and how are they related to each other?

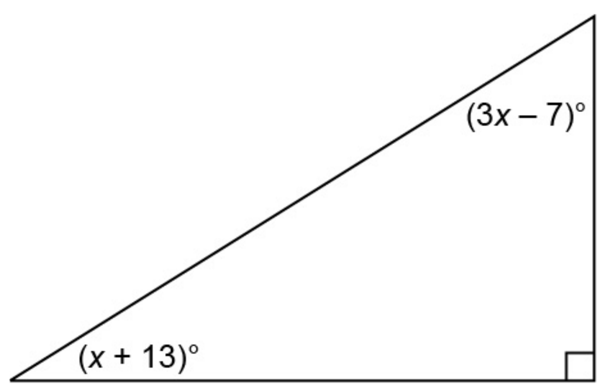
M.P.7. Look for and make use of structure. Know that two angles that can be put together to form a right angle are called complementary and that their angle measures sum to 90°. For example, if a 90° angle is divided into two angles, $x°$ and $(6x - 1)°$, then x can be determined by solving the equation $x + (6x - 1) = 90$, and the complementary angle, $(6x - 1)°$, can be found by substituting in the solution for x. Additionally, because the sum of angles in a triangle is 180°, the sum of the two acute angles in a right triangle is 90°, making them complementary.

- Ask students to determine the measure of the complement of a given angle. For example, given an angle measure of 23°, the measure of the complement is determined to be 67° by finding the difference of 90° and 23°.

- Ask students to verify that the acute angles in a right triangle are complementary. For example, given a right triangle with one angle measuring 32°, determine the unknown angle measure to be 58° by subtracting the sum of 90° and 32° from 180°, then verifying that the sum of 32° and 58° is 90°.



- Ask students to determine the measures of angles in a figure by writing and solving an equation with complementary angles. For example, give students the figure shown and ask them to find the measures of the two acute angles.

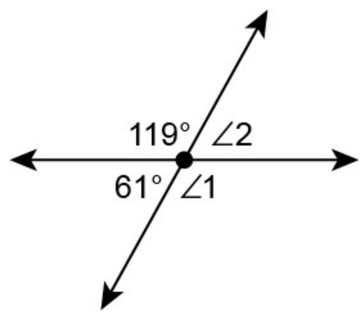


The sum of the three angles is equal to 180°, thus the measures of the acute angles can be found by solving the equation $(3x - 7) + (x + 13) = 90$ to get a solution of $x = 21$. This value can be substituted back into each angle expression to get angle measures of 34° and 56°.

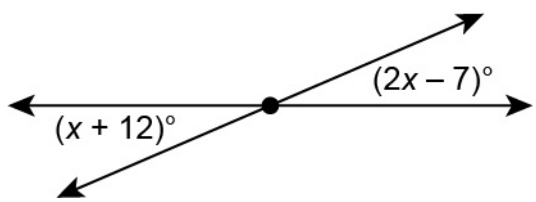
What are vertical angles and how are they related to each other?

M.P.7. Look for and make use of structure. Explain that when two lines intersect, the angles that are formed opposite each other are called vertical angles, and their angle measures are equal. For example, if two lines intersect to create a 43° angle, they also create a 137° angle. Since the angle that is opposite the 43° angle is supplementary to the 137° angle, it must also be 43°. Additionally, the two pairs of adjacent vertical angles have a sum of 360° because two sets of supplementary angles are formed.

- Ask students to determine the angle measures of vertical angles by subtracting known angle measurements from 180°. For example, in the following figure, the measure of angle 1 must be 119° because it is supplementary to the 61° angle, and the measure of angle 2 must be 61° because it is supplementary to the 119° angle.



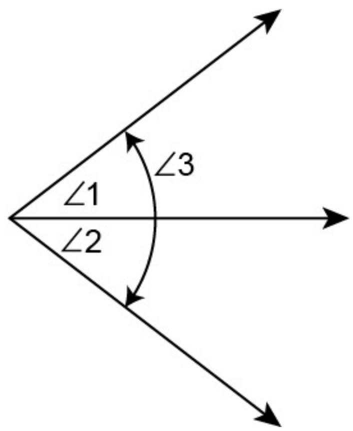
- Ask students to determine measures of vertical angles by creating and solving an equation. If two vertical angles measure $(x + 12)°$ and $(2x - 7)°$, then x can be determined by solving the equation $x + 12 = 2x - 7$. The measures of the two angles can then be found by substituting the solution of $x = 19$ into each expression and calculating a measure of 31°.



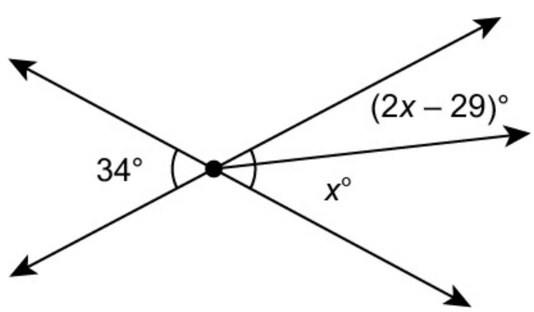
What are adjacent angles and how are they related to each other?

M.P.7. Look for and make use of structure. Know that two angles that share a common vertex and a common side are called adjacent, and use the additive nature of angles to solve problems. For example, given two adjacent angles of $m°$ and $(5m + 9)°$ that have a supplementary angle of 75° in a figure, m can be determined by writing the equation $m + (5m + 9) + 75 = 180$. Additionally, an angle of 40° can be bisected, creating two adjacent angles each measuring 20°.

- Ask students to write equations that express the measure of an angle as a sum or difference of other measures. For example, students can describe the relationship between the angles shown by writing a sum, such as $\mathrm{m}\angle 1 + \mathrm{m}\angle 2 = \mathrm{m}\angle 3$, or as a difference, such as $\mathrm{m}\angle 3 - \mathrm{m}\angle 2 = \mathrm{m}\angle 1$.



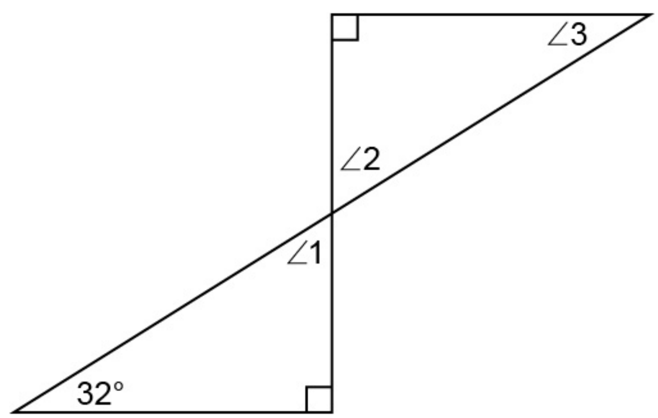
- Ask students to determine the measures of angles in a figure by writing and solving an equation that relates the adjacent angles to one another or to a third angle. For example, the measures of angles $x°$ and $(2x - 29)°$ can be determined by equating their sum to the vertical angle 34° to create the equation $x + (2x - 29) = 34$. The equation can be solved to attain a solution of $x = 21$, which can be substituted into each angle expression to get angle measures of 21° and 13°.



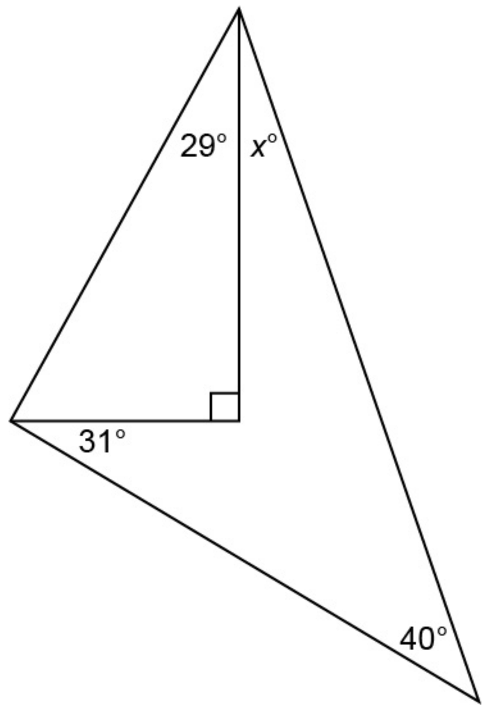
How can the relationships between angles in a figure help to solve problems?

M.P.2. Reason abstractly and quantitatively. Observe a figure to find how an unknown angle measure relates to the angle measures given. For example, given a figure with several angles, use what is known about complementary angles and vertical angles to find a missing angle measure. Additionally, given a figure with several angles, use what is known about adjacent angles and supplementary angles to find a missing angle measure.

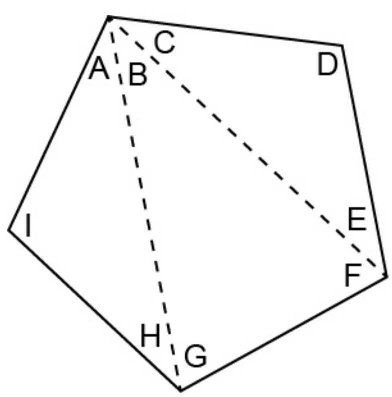
- Ask students to use vertical angles to determine unknown measurements in a figure. In the example shown, the measure of angle 1 is 58° because it is complementary to the 32° angle, the measure of angle 2 is also 58° because angle 1 and angle 2 are a pair of vertical angles, and the measure of angle 3 is 32° because it is complementary to angle 2.



- Ask students to use complementary and adjacent angles to determine unknown angle measurements in a figure. In the example shown, first use the complementary angle relationship to find the missing value in the right triangle, add together the known values of the adjacent angles on the left side of the larger triangle, and then use the fact that angles in a triangle total a sum of 180° to find x. The value of x is determined to be 19° because 29° is complementary to 61° in the right triangle, and the measure of 19° is needed for the angles in the large triangle to sum to 180°.



- Ask students to determine the measure of interior angles in a polygon using adjacent angles. For example, given a regular pentagon, connect vertices with two lines to form three triangles, whose angles have a sum of $3 \times 180° = 540°$ (in the image shown, the sum $A + H + I = 180°$, $B + F + G = 180°$, and $C + D + E = 180°$), meaning that each interior angle in the regular pentagon measures $540° \div 5 = 108°$.



Key Academic Terms:

supplementary angles, complementary angles, vertical angles, adjacent angles, straight angle, angle measure, right angle, congruent

Additional Resources:

- Video: Finding unknown angles
- Video: Thinkport | Mapping sculptures
- Lesson: Angles everywhere

22

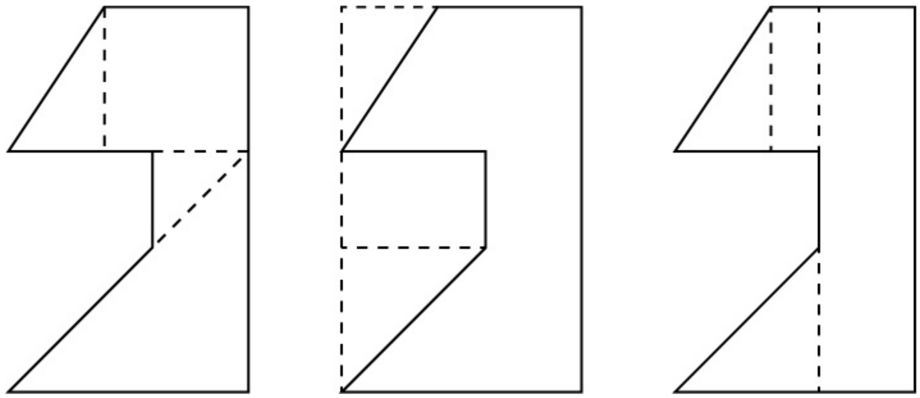
Geometry and Measurement
Solve real-world and mathematical problems involving angle measure, circumference, area, surface area, and volume. *Note: Students must select and use the appropriate unit for the attribute being measured when determining length, area, angle, time, or volume.*
22. Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right rectangular prisms.

Guiding Questions with Connections to Mathematical Practices:

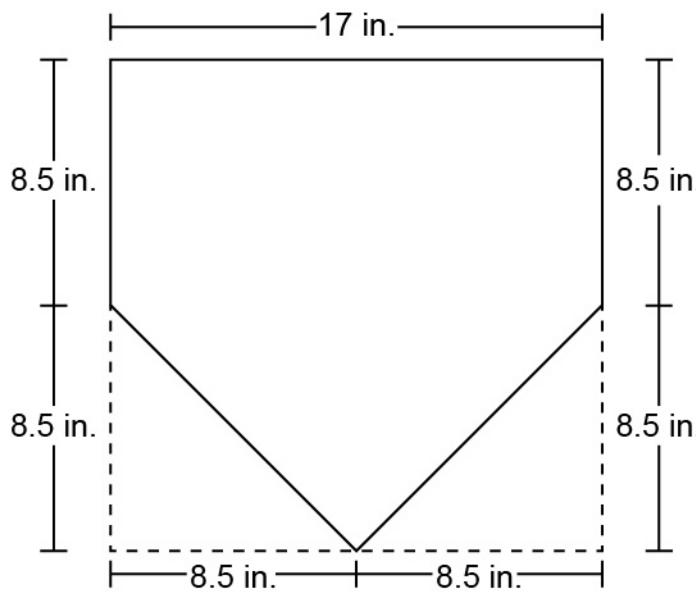
How can two-dimensional objects be decomposed to help solve problems involving area?

M.P.1. Make sense of problems and persevere in solving them. Represent any two-dimensional figure as being composed of triangles, quadrilaterals, and other polygons to find the area. For example, given a rectangular lawn that is 25 feet long and 17 feet wide with a concrete square patio with side lengths of 7 feet in the middle, determine the number of square feet of sod needed to cover the lawn without covering the patio. This can be solved by first finding the area of the lawn (25×17) and then subtracting the area of the patio (7×7). Additionally, the area can be found using a sum of four rectangles of lawn, two of which are (25×5) and two of which are (9×7).

- Ask students to show or explain multiple strategies for determining the area of a two-dimensional figure that is composed of multiple polygons. For example, the area of the figure shown can be determined by finding the sum of three triangles and a square, OR by subtracting the area of two triangles and a rectangle from a larger rectangle, OR by finding the sum of the areas of two rectangles and two triangles.



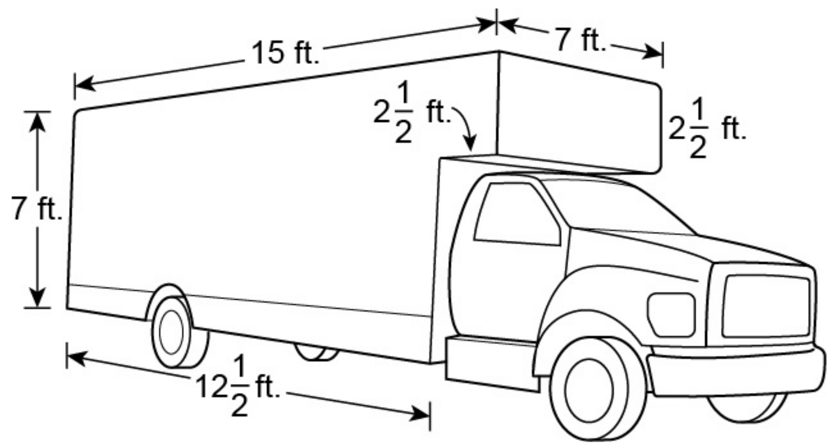
- Ask students to determine the area of a two-dimensional figure that is composed of multiple polygons. For example, the image shows the approximate dimensions of home plate at a major league baseball field. The triangles removed from the larger square each have legs of 8.5 inches, so the area of home plate can be determined by subtracting the areas of the two triangles from the square: (17 × 17) – (0.5 × 8.5 × 8.5) – (0.5 × 8.5 × 8.5) square inches.



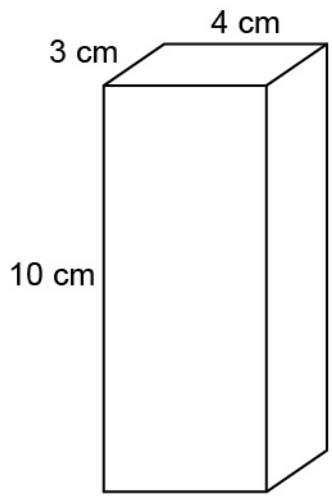
How can three-dimensional objects be decomposed to help solve problems involving surface area or volume?

M.P.7. Look for and make use of structure. Decompose a three-dimensional object to see the two-dimensional shapes that compose it to find surface area and volume. For example, a right rectangular prism is composed of 6 faces, all of which are rectangles, so the surface area will be the sum of the areas of those rectangles. The volume will be the area of a special cross section, known as the base, multiplied by the height of the prism. Additionally, the volume of a three-dimensional object composed of multiple prisms can be determined using the sum of the volumes of the individual prisms.

- Ask students to determine the volume of a solid composed of two or more rectangular prisms. For example, a moving truck has an internal width of 7 feet, a height of 7 feet, and a length of 12.5 feet in the main compartment and a height of 2.5 feet and length of 2.5 feet in an additional compartment above the cab. The volume of the entire truck can be determined by the sum of $7 \times 7 \times 12.5$ and $7 \times 2.5 \times 2.5$ cubic feet.



- Ask students to determine the surface area of a right rectangular prism with given dimensions. For example, a prism has a height of 10 centimeters and a base that measures 3 centimeters by 4 centimeters. The surface area is the sum of the areas of six rectangles: $(3 \times 4) + (3 \times 4) + (3 \times 10) + (3 \times 10) + (4 \times 10) + (4 \times 10) = 164$ square centimeters.



Key Academic Terms:

two-dimensional, three-dimensional, area, volume, surface area, prism, base

Additional Resources:

- Lesson: Surface area of prisms
- Activity: Sand under the swing set
- Activity: Surface area and volume
- Lesson: Volume of square pyramids
- Lesson: Write-It Wednesday surface area